TUTORIAL 7

Date :

(2) Show that if X & Y are connected, then $X \times Y$ is connected

$X \simeq X \times \{y_0\}$, let $h_{(x_0, y_0)} = X \times \{y_0\} \cup \{x_0\} \times Y$
$Y \simeq \{x_0\} \times Y$

conn. since X, Y conn & $X \times \{y_0\} \cap \{x_0\} \times Y = \{(x_0, y_0)\}$

$X \times Y = \bigcup_{(x_0, y_0) \in X \times Y} h_{(x_0, y_0)}$ is conn. because each $h_{(x_0, y_0)}$ is conn & $h_{(x_0, y_0)} \cap h_{(x_1, y_1)} \neq \emptyset$

(3) Which of the following are homeomorphic?

$(0,1) \simeq \mathbb{R}$, $f(x) = \dfrac{1}{1+e^{-x}}$

$(0,1] \simeq [1, \infty)$, $f(x) = 1/x$

We just need to show $(0,1)$, $(0,1]$ & $[0,1]$ are not homeomorphic.

① $(0,1) \not\simeq [0,1]$

Suppose $\exists$ homeomp $g : [0,1] \to (0,1)$

$\Rightarrow g|_{(0,1]} : (0,1] \to (0,1) \setminus \{g(0)\}$ is homeomp

$\Rightarrow g|_{(0,1)} : \underbrace{(0,1)}_{\text{conn}} \to \underbrace{(0,1) \setminus \{g(0), g(1)\}}_{\text{disconn}}$ is homeomp.

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Contd^n

$\Rightarrow [0,1] \not\cong (0,1)$

(2) $[0,1] \not\cong (0,1]$

Suppose $\exists$ homeomp $g : [0,1] \to (0,1]$

$\Rightarrow g|_{(0,1)} : (0,1) \to (0,1] \setminus \{g(1)\}$ is homeomp

$\Rightarrow g|_{(0,1)} : \underbrace{(0,1)}_{\text{conn.}} \to \underbrace{(0,1] \setminus \{g(0), g(1)\}}_{\text{discon.}}$ is homeomp

Because, if $g(1)=1$, $g(0)=b \Rightarrow (0,1]\setminus\{g(0),g(1)\} = (0,b) \cup (b,1)$

else, $g(0), g(1) \neq 1$, $\Rightarrow (0,a) \cup (a,b) \cup (b,1]$

$g(0)=a < g(1)=b$ say

Contd^n

$\Rightarrow [0,1] \not\cong (0,1]$

(3) $(0,1] \not\cong (0,1)$

Same argument as (1). Remove $g(1)$ from $(0,1)$.

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(4) Are $\mathbb{R}$ & $\mathbb{R}^2$ homeomorphic ?

No.

Suppose $\exists$ homeomp. $g : \mathbb{R}^2 \to \mathbb{R}$

$\Rightarrow g|_{\mathbb{R}^2 \setminus \{(0,0)\}} : \underbrace{\mathbb{R}^2 \setminus \{(0,0)\}}_{\text{conn.}} \to \underbrace{\mathbb{R} \setminus \{g(0,0)\}}_{\text{disconn.}}$

$= (-\infty, a) \cup (a, \infty)$

where $a = g(0,0)$

Consider $L_m = S^1 \cup \{(x, mx) : x \in \mathbb{R} \setminus \{0\}\}$

$L_* = S^1 \cup \{(0,y) : y \in \mathbb{R} \setminus \{0\}\}$

Each L_m & L_* is conn. & pairwise intersecting.

& $\mathbb{R}^2 \setminus \{0\} = \bigcup_{m \in \mathbb{R}} L_m \cup L_*$

So, $\mathbb{R}^2 \setminus \{0\}$ is conn.

(5) Show that the space obtained on removing $[0,1] \times \{1/2\}$ in the Mobius strip is homeomophic to a cylinder

Let $M = \dfrac{[0,1] \times [0,1]}{(0,y) \sim (1, 1-y)}$, $A = [0,1] \times \{1/2\}$

Let $X = [0,1] \times [0,1] \setminus A = [0,1] \times ([0, 1/2) \cup (1/2, 1])$

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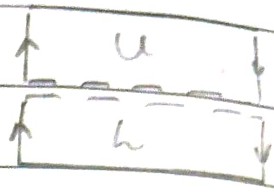
Imposing $(0,y) \sim (1,1-y)$ on X:

$y < 1/2 \to 1-y > 1/2$

& $y > 1/2 \Rightarrow 1-y < 1/2$



So, left edge of L glues to right edge of U
& left edge of U glues to right edge of L.

To reparametrize, define a new vertical coordinate $t \in (0,1)$

$$t = \begin{cases} 2y, & y < 1/2 \\ 2(1-y) & y > 1/2 \end{cases}$$

Under $(0,y) \sim (1,1-y)$:

If $y < 1/2$, $t(0,y) = 2y$

If $y > 1/2$, $t(1,1-y) = 2(1-(1-y)) = 2y$

Hence t is preserved under the identification

Define a map $F : X \to S^1 \times (0,1)$

$$F(x,y) = (e^{2\pi i x}, t(y))$$

We will check if $F(0,y) = F(1,1-y)$

① $e^{2\pi i \cdot 0} = 1 = e^{2\pi i \cdot 1}$

② Already checked $t(0,y) = t(1,1-y)$

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Hence, F descends to a quotient map

$$\tilde{F} : X/\sim \;\to S^1 \times (0,1)$$

<u>Continuity</u> : F is cts! (cts. in each coordinate)

$\Rightarrow \tilde{F}$ is cts. (universal ppt of quotients)

<u>Bijectivity</u> :

- <u>Injectivity</u> : Suppose $\tilde{F}([x_1, y_1]) = \tilde{F}([x_2, y_2])$

$\Rightarrow e^{2\pi i x_1} = e^{2\pi i x_2} \Rightarrow x_1 = x_2$ or $|x_1 - x_2| = 1$ endpts.

& $t(y_1) = t(y_2) \Rightarrow y_1 = y_2$ or $y_1 = 1 - y_2$

$\Rightarrow (x_1, y_1) \sim (x_2, y_2)$

$\Rightarrow [x_1, y_1] = [x_2, y_2]$

- <u>Surjectivity</u> : Take any $(e^{2\pi i x}, t) \in S^1 \times (0,1)$

Choose $y = t/2$! Then $t(y) = t$. So, $\tilde{F}([x, y]) = (e^{2\pi i x}, t)$

<u>Inverse</u> : $\tilde{G} : S^1 \times (0,1) \to X/\sim$

$$\tilde{G}(e^{2\pi i x}, t) = [x, t/2]$$

$\tilde{G}$ is cts. & $\tilde{G} \circ \tilde{F} = id$ & $\tilde{F} \circ \tilde{G} = id$.

Hence, $\tilde{F}$ is a homeomorph. & thus $X/\sim \;\simeq S^1 \times (0,1)$

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(6) Consider $H = \{(x,y) \in \mathbb{R}^2 : y \geq 0\}$ & let
$H_+ = \{(x,y) \in \mathbb{R}^2 : y > 0\}$ with topology generated by
base $\{N((a,b), \delta) \cap H_+ \mid (a,b) \in H_+\} \cup \{N((a,0), \delta) \cap H_+ \cup \{(a,0)\} : a \in \mathbb{R}\}$
This is called Moore topology & is denoted by H_M.

Show $\mathbb{R} \times \{0\}$ is a closed subset of H_M

We will show complement of $\mathbb{R} \setminus \{0\}$ is open.

$$H \setminus (\mathbb{R} \setminus \{0\}) = H_+$$

TST: H_+ is open in Moore topology.

Let $(x,y) \in H_+ \Rightarrow y > 0$.

By definition of basis, for any $\delta > 0$, $N((x,y), \delta) \cap H_+$ is a basic open set. Moreover, this set is contained in H_+

$\Rightarrow H_+$ is open in H_M

$\Rightarrow \mathbb{R} \setminus \{0\}$ is closed subset of H_M.

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(7) Show that $S = \{(x, -x) : x \in \mathbb{R}\}$ is a closed subset of $\mathbb{R}_\ell \times \mathbb{R}_\ell$ with product topology.

We will show complement of S is open.

Let $(a, b) \notin S \Rightarrow a+b \neq 0$

So, either $a+b>0$ or $a+b<0$

Case 1: $a+b>0$.

Choose $\epsilon>0$. Let $U = [a, a+\epsilon) \times [b, b+\epsilon)$

For any $(x, y) \in U$, $x+y \geq a+b > 0 \Rightarrow U \cap S = \emptyset$

Case 2: $a+b<0$

Choose $\epsilon>0$ s.t $(a+b)+2\epsilon<0$

Let $U = [a, a+\epsilon) \times [b, b+\epsilon)$

For any $(x, y) \in U$, $x+y < a+b+2\epsilon < 0 \Rightarrow U \cap S = \emptyset$

Hence, we found a basis nbd of U s.t $U \subseteq \mathbb{R}_\ell \times \mathbb{R}_\ell \setminus S$

$\Rightarrow S$ is closed.

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(8) Show that $X = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \in \mathbb{Q} \text{ or } x_2 \in \mathbb{Q}\}$ is path connected & hence connected.

Consider (x_1, y_1) & $(x_2, y_2) \in X$.

WLOG, we can consider the following two cases

① $x_1, x_2 \in \mathbb{Q}$

② $x_1 \in \mathbb{Q}$, $y_2 \in \mathbb{Q}$

The cases $y_1, y_2 \in \mathbb{Q}$ & $x_2, y_1 \in \mathbb{Q}$ can be proved similarly to case 1 & 2 resp.

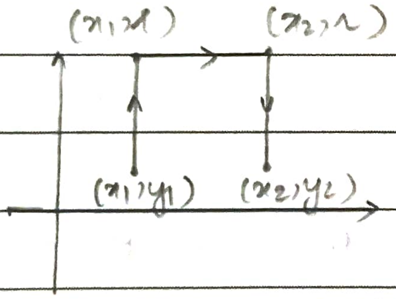
Case 1 : $x_1, x_2 \in \mathbb{Q}$

Fix $r \in \mathbb{Q}$. Then consider paths $\gamma_1(t) = (x_1, (1-t)y_1 + tr)$

$\gamma_2(t) = ((1-t)x_1 + tx_2, r)$

$\gamma_3(t) = (x_2, (1-t)r + ty_2)$

$\forall\, t \in [0,1]$



$\because$ $x_1 \in \mathbb{Q} \Rightarrow \gamma_1(t) \subseteq X$

$r \in \mathbb{Q} \Rightarrow \gamma_2(t) \subseteq X$

$x_2 \in \mathbb{Q} \Rightarrow \gamma_3(t) \subseteq X$

The concatenation of $\gamma_1, \gamma_2, \gamma_3$ is a path in X connecting (x_1, y_1) & (x_2, y_2)

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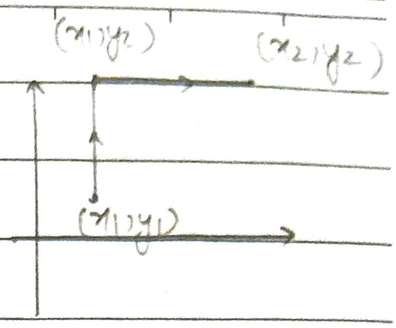
Case 2: $x_1 \in Q, \ y_2 \in Q$



Consider $\gamma_1(t) = (x_1, (1-t)y_1 + ty_2)$

$\gamma_2(t) = ((1-t)x_1 + tx_2, y_2)$

$\forall \ t \in [0,1]$

$\because x_1 \in Q \Rightarrow \gamma_1(t) \subseteq X$

$y_2 \in Q \Rightarrow \gamma_2(t) \subseteq X$

The concatenation of γ_1 & γ_2 is a path in X connecting (x_1, y_1) & (x_2, y_2)

Since for any two pts in X, we can construct a path joining them.
Therefore, X is path conn., hence conn.

(9) A topo sp is said to be second countable if it has a countable base.

a) Subspace of countable space is countable. Prove or disprove
b) A finite product of second countable spaces is second countable. Prove or disprove
c) Cts image of a second countable space is second countable. Prove or disprove.

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a) Yes.

Subspace of a second countable space is second countable

Let X be a topo. sp. which is second countable.

Then $\exists$ countable basis $\mathcal{B} = \{B_1, B_2, \dots\}$

Let $Y \subseteq X$ with subspace topology.

By def^n, open sets in Y are of the form $U \cap Y$ where U is open in X.

Claim : $\mathcal{B}_Y = \{B_k \cap Y : B_k \in \mathcal{B}\}$ is a countable basis of Y.

Since $\mathcal{B}$ is countable, $\mathcal{B}_Y$ is countable.

For any open set $U \subseteq Y$, $\exists$ open set $V \subseteq X$ s.t. $U = V \cap Y$

Since $\mathcal{B}$ is a basis of X, $V = \bigcup B_i \Rightarrow U = \bigcup (B_i \cap Y)$

Hence $\mathcal{B}_Y$ is a basis.

b) True.

Let X & Y be second countable space with basis $\mathcal{B}_X = \{B_1, B_2, \dots\}$ & $\mathcal{B}_Y = \{C_1, C_2, \dots\}$

The standard basis for product topology on $X \times Y$ is

$$\mathcal{B} = \{B_i \times C_i : B_i \in \mathcal{B}_X, C_i \in \mathcal{B}_Y\}$$

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Since $\mathcal{B}_X$ & $\mathcal{B}_Y$ are countable, so is $\mathcal{B}$.

Hence, $X \times Y$ is second countable

By induction we can extend it to all finite products.

Infinite product: Not necessarily second countable.

eg: $\mathbb{R}^{\mathbb{N}}$.

c) No.

Consider identity map $f: \mathbb{R} \to (\mathbb{R}, \text{cofinite})$

① f is cts.

Let $U \subseteq (\mathbb{R}, \text{cofinite})$

If $U = \emptyset$, $f^{-1}(U) = \emptyset \to$ open in $\mathbb{R}$.

Else $\mathbb{R} \setminus U = \{x_1, \dots, x_n\} \Rightarrow U = \mathbb{R} \setminus \{x_1, \dots, x_n\}$

$\Rightarrow f^{-1}(U) = \mathbb{R} \setminus \{x_1, \dots, x_n\} \to$ open in $\mathbb{R}$

So, f is cts

② $(\mathbb{R}, \text{cofinite})$ is not second countable

Suppose $\mathcal{B}$ is a countable basis of $(\mathbb{R}, \text{cof})$. Then for every $U \in \mathcal{B}$, $\mathbb{R} \setminus U$ is finite. Take $\bigcup_{U \in \mathcal{B}} (\mathbb{R} \setminus U)$. Then $\bigcup_{U \in \mathcal{B}} (\mathbb{R} \setminus U)$ is countable

(countable union of finite sets)

Pick $x \in \mathbb{R} \setminus \bigcup_{U \in \mathcal{B}} (\mathbb{R} \setminus U)$.

Then $\mathbb{R} \setminus \{x\}$ is open in cofinite topology, but can't be written as a union of sets in $\mathcal{B}$

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This is because if $x \in \mathbb{R} \setminus \bigcup_i (\mathbb{R} \setminus U_i) \Rightarrow x \notin U_i^c \; \forall i$

$\Rightarrow x \in U_i \; \forall i$

But if $\mathbb{R} \setminus \{x\} = \bigcup_i U_i \Rightarrow x \notin U_i$ for some i

Contdn.

Hence ($\mathbb{R}$, cofinite) is not second countable

(10) A topo. sp. is separable if it has a countable dense subset

a) Show $\mathbb{R}$ with std. topo. is separable

b) $\mathbb{R}^n$ with std. topo is separable

c) H_M is separable but its subspace $\mathbb{R} \setminus \{0\}$ is not.

a) Consider subset $\mathbb{Q} \subseteq \mathbb{R}$.

$\mathbb{Q}$ is countable.

In std. topo., basis open sets are intervals.

So, it suffices to check, for any $a<b$, $(a,b) \cap \mathbb{Q} \neq \emptyset$

Since for any $a, b \in \mathbb{R}$, $\exists \; q \in \mathbb{Q}$ s.t $a<q<b \Rightarrow (a,b) \cap \mathbb{Q} \neq \emptyset$

Hence $\mathbb{Q}$ is dense in $\mathbb{R}$.

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b) Consider $\mathbb{Q}^n = \{(q_1, \dots, q_n) : q_i \in \mathbb{Q}\}$

Since $\mathbb{Q}$ is countable & finite product of countable sets is countable, so $\mathbb{Q}^n$ is countable.

In std. topo, basic open sets are product of intervals.
So, it suffices to check, for any $U = (a_1, b_1) \times \dots \times (a_n, b_n)$,
$U \cap \mathbb{Q}^n \neq \emptyset$.

Since for any $a_i, b_i \in \mathbb{R}$, $\exists\, q_i \in \mathbb{Q}$ s.t. $a_i < q_i < b_i$
$\Rightarrow (q_1, q_2, \dots, q_n) \in U \cap \mathbb{Q}^n$.

Hence, $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$.

c) Consider $D = \mathbb{Q}^2 \cap \{y > 0\}$

$\mathbb{Q}^2$ is countable $\Rightarrow D$ is countable

We must show every non-empty basic open set intersects D.

Case 1 : Open set around a pt. with $y > 0$.
These are usual open sets of $\mathbb{R}^2$. Since $\mathbb{Q}^2$ is dense in $\mathbb{R}^2$, such sets contain pts. of D.

Case 2 : Neighbourhood of b'ry pt. $(x, 0)$

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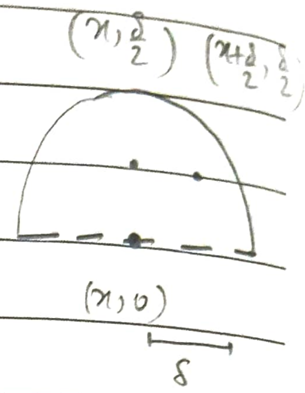
It is given by $(N((x,0),\delta) \cap H_+) \cup \{(x,0)\}$



Then the pt. $(x+\frac{\delta}{2},\frac{\delta}{2})$ is contained in the nbd.

So, $\exists$ $q_1, q_2 \in \mathbb{Q}$ s.t $x < q_1 < x+\frac{\delta}{2}$,

$0 < q_2 < \frac{\delta}{2}$

s.t $(q_1, q_2) \in D$

$\Rightarrow (q_1,q_2) \in D \cap (\text{nbd of } (x,0)) \neq \emptyset$

Hence D is dense in H_M.

Now, we will show $\mathbb{R}\times\{0\}$ is <u>not</u> separable.

Under the subspace topo., $\{(x,0)\}$ is open in $\mathbb{R}\times\{0\}$.

($\because \{(x,0)\} = \mathbb{R}\times\{0\} \cap$ (nbd of $(x,0)$ in H_M))

So, $\mathbb{R}\times\{0\}$ is discrete.

In a discrete space, set is dense <u>iff</u> it equals the whole space.

($\because$ Closure of each pt. is itself.)

But $\mathbb{R}\times\{0\}$ is uncountable.

Hence, $\nexists$ countable dense subset, so $\mathbb{R}\times\{0\}$ is <u>not</u> separable.

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(1) Let $V = \{(x,0) \in \mathbb{R}^2 : x \leq 0\} \cup \{(x, \sin(1/x)) : x > 0\}$
be the topologist's sine curve.
Let $T = \{(x, \sin(1/x)) : x > 0\}$

a) Show V is conn.
b) Show T is path conn.
c) Show V is <u>not</u> path conn.
d) Closure of a path conn. space is path conn.
Prove or disprove

a) Let $A = \{(0,0)\} \cup \{(x, \sin(\frac{1}{x})) : x > 0\}$
$\overline{T} = \{(x, \sin(\frac{1}{x})) : x > 0\} \cup \{0\} \times [-1, 1]$

Note, $T \subseteq A \subseteq \overline{T}$

Now, T is conn. as it is cts. img of conn.

$$(0, \infty) \longrightarrow \mathbb{R}^2$$
$$x \longmapsto (x, \sin(\tfrac{1}{x}))$$

$\Rightarrow$ A is conn. $\quad (\underbrace{T}_{\text{conn.}} \subseteq A \subseteq \underbrace{\overline{T}}_{\text{conn.}})$

Now, $V = \{(x,0) \in \mathbb{R}^2 : x < 0\} \cup A$ $\rightarrow$ pairwise intersecting & conn.
$\Rightarrow$ V is conn.

$$\left\{ \begin{array}{c} \because \text{cts. img of} \\ (-\infty, 0) \longrightarrow \mathbb{R}^2 \\ x \longmapsto (x, 0) \end{array} \right\}$$

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b) Let $(x, \sin(\frac{1}{x}))$, $(y, \sin(\frac{1}{y}))$

Then $\gamma(t) = \left(x(1-t)+yt,\ \sin\left(\frac{1}{x(1-t)+yt}\right)\right) \subseteq T$

is a path connecting the two pts.

c) Claim : There is no path joining $(0,0)$ & $(\frac{1}{\pi}, 0)$

Let $\exists$ a path in V joining $(0,0)$ & $(\frac{1}{\pi}, 0)$

i.e $v : [0,1] \to V$ a cts. fn^n $v(t) = (v_1(t), v_2(t))$

Now, $v_1(t)$ is cts. ($\because$ proj into $\mathbb{R}$ is) & for all $v_1(t) > 0$,

$v_2(t) = \frac{1}{\sin(v_1(t))}$

Since v_1 is cts. & $v_1(t)$ takes values b/w 0 & $1/\pi$,

for any seq $t_n \to 0$, $v_1(t_n) \to 0$

Then for any $n > 0$, $\exists\ k_n \in \mathbb{Z}_{>0}$ s.t $0 < \frac{2}{\pi k_n} < v_1(t_n)$

& $\exists\ 0 \le s_n \le t_n$ s.t $v_1(s_n) = \frac{2}{\pi k_n}$

using intermediate value thm.

Thus, we have $s_n \to 0$ & $v(s_n) = \left(v_1(s_n), \sin\left(\frac{1}{v_1(s_n)}\right)\right) \to (0,0) = v(0)$

$= \left(\frac{2}{\pi k_n}, \pm 1\right) \not\to (0,0)$

which is a contd^n.

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d) False

$T = \{ (x, \sin(\frac{1}{x})) : x > 0 \}$

$\overline{T} = \{ (x, \sin(\frac{1}{x})) : x > 0 \} \cup \{0\} \times [-1, 1]$

By the same argument as (c), $\overline{T}$ is not path conn.

(2) Prove or disprove

a) Cts. img. of path conn. sp. is path conn.

b) Open img. of path conn. sp. is path conn.

c) Product of finitely many path conn. sp. is path conn.

a) True

Let X be path conn. & $f : X \to Y$ be cts.

Then for any $x, y \in f(X)$, $\exists\, a, b \in X$ s.t $x = f(a)$ & $y = f(b)$

Since X is path conn., $\exists$ cts $\gamma : [0,1] \to f(X)$ s.t $\gamma(0) = a$, $\gamma(1) = b$

Then $f \circ \gamma : [0,1] \to Y$ s.t $f \circ \gamma(0) = f(\gamma(0)) = f(a) = x$
$f \circ \gamma(1) = f(\gamma(1)) = f(b) = y$

Since f, γ cts. $\Rightarrow f \circ \gamma$ cts.

Hence, $f \circ \gamma$ is the req. path in $f(X)$ joining x & y.

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b) False

Consider $f : \mathbb{R} \to \{0,1\}$
usual topo (on $\mathbb{R}$), discrete topo (on $\{0,1\}$).

$$f(x) = \begin{cases} 0, & x \in \mathbb{Q} \\ 1, & x \notin \mathbb{Q} \end{cases}$$

① f is open map
For any open set $U \subset \mathbb{R}$, both $U \cap \mathbb{Q} \neq \emptyset$ & $U \cap (\mathbb{R} \setminus \mathbb{Q}) \neq \emptyset$
$\Rightarrow f(U) = \{0,1\}$, which is open

So, f is open map

② Image is not path conn.
$\{0,1\} = \{0\} \cup \{1\}$ $\Rightarrow$ $\{0,1\}$ is disconn.
open, open $\Rightarrow$ $f(\mathbb{R})$ is not path conn.

c) True
Let $X_1, X_2, \ldots, X_n$ be path-conn. spaces & consider
$$X = X_1 \times \ldots \times X_n$$

Consider $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in X$
Since each X_i is path conn., $\exists$ path $\gamma_i : [0,1] \to X_i$
$\gamma_i(0) = x_i$
$\gamma_i(1) = y_i \quad \forall 1 \le i \le n$

Now, def. $\gamma(t) = (\gamma_1(t), \gamma_2(t), \ldots, \gamma_n(t))$
Each coordinate is cts. $\Rightarrow \gamma$ is cts. & $\gamma(0) = x$, $\gamma(1) = y$
So, this is a path in X. Hence, X is path conn.

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(3) Show that $\mathbb{R}_\ell$ is not locally conn.

For $\mathbb{R}_\ell$ to be locally conn, for any $x \in \mathbb{R}_\ell$ & open U s.t $x \in U$, $\exists$ conn. open set W s.t $x \in W \subseteq U$.

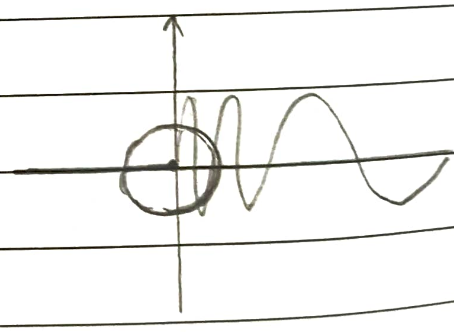
However, every open set W in $\mathbb{R}_\ell$ is disconn.
We show this by showing basic open sets in $\mathbb{R}_\ell$ are disconn.

$$[a,b) = [a,c) \cup [c,b)$$

Hence, $\mathbb{R}_\ell$ is not locally conn.

(4) Show that V is not locally conn

Let $U = N((0,0),r) \cap V$; $0 < r < 1$
U is open nbd. of $(0,0)$ in subsp. topo.



For $n > \frac{1}{r\pi}$, $n \in \mathbb{Z}_{>0}$, consider intervals when

$$\frac{1}{x} \in (n\pi, (n+1)\pi)$$

These pieces are separated from each other inside V.

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Suppose $\exists$ conn. open set $W \subset V$ with $(0,0) \in W \subset V$

Then W must intersect infinitely many oscillating arcs.

But these arcs are separated in V.

So, W would split into disjoint non-empty open subsets $\to$ Contd^n

Hence no nbd of $(0,0)$ contains a conn. open nbd.

& thus V is not locally conn.

(5) Prove or disprove

a) Cts img. of locally conn. sp. is locally conn

b) Product of finitely many locally conn. spaces is locally conn

c) Infinite product of locally conn. spaces where all but finitely many are conn. is locally conn.

a) False

$id : \mathbb{R} \to \mathbb{R}_\ell$ (discrete)

id is cts.

$\mathbb{R}$ (discrete) is locally conn.

$\mathbb{R}_\ell$ is not locally conn.

b) True

Let $X_1, \dots, X_n$ be loc. conn. sp. & $X = X_1 \times \dots \times X_n$

Let $x = (x_1, \dots, x_n) \in X$ & let $U \subset X$ be an open set containing x.

Then by def^n of product top, $\exists$ basic open set

$U_1 \times \dots \times U_n \subset U$ with $x_i \in U_i$, each U_i open in X_i

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Since each X_i is loc. conn., for each i, $\exists$ conn. open set

$$V_i \subset U_i \;,\; x_i \in V_i$$

Def. $V = V_1 \times \dots \times V_n$

V is open in product topo., $x \in V \subset U$

Each V_i is conn. $\Rightarrow$ finite prod is conn.

Hence, X is loc. conn.

c) True

Let $\{X_i\}_{i\in J}$ be a family of spaces s.t

- each X_i is loc. conn.
- all but finitely many are conn.

Consider $X = \prod_{i\in J} X_i$

Let $x = (x_i) \in X$ & let $U \subset X$ be an open set containing x

Then $\exists$ basic open set $\prod_{i\in J} U_i$ where

$$x \in \prod_{i\in J} U_i \;,\; \text{each } U_i \text{ open in } X_i$$
$$\& \; U_i = X_i \text{ for all but finitely many } i.$$

Let $F = \{i : U_i \neq X_i\}$. F is finite.

For each $i \in F$, since X_i is loc. conn., $\exists$ conn. open set

$$V_i \subset U_i \;,\; x_i \in V_i$$

Date :

Define $V_i = \begin{cases} \text{conn. open subset of } U_i & , i \in F \\ X_i & , i \notin F \end{cases}$

& let $V = \prod_{i \in J} V_i$

Then V_i is open & $x \in V \subset U$

Moreover, $V = \underbrace{\left(\prod_{i \in F} V_i\right)}_{\substack{\text{finite product} \\ \text{of conn. spaces} \\ = \text{Conn.}}} \times \underbrace{\left(\prod_{i \notin F} X_i\right)}_{\substack{\text{prod. of conn. sp.} \\ \Rightarrow \text{conn.}}}$

Thus V is conn.

Hence, X is loc. conn.

(6) Any fx^n from a compact space to a Hausdorff space is closed

Statement is only true for cts. fx^{n}s.

Closed subset of compact space is compact.

Cts. image of compact set is compact.

Compact subset of Hausdorff space is closed.

Hence, any cts. fx^n from compact sp. to Hausdorff sp. is closed.

Date : ____________

(7) Let $\mathcal{T}$ & $\mathcal{U}$ be topo. on X s.t $\mathcal{T} \subseteq \mathcal{U}$.
If X is compact with $\mathcal{T}$ topology, it is compact with $\mathcal{U}$ topology? What about other way around?

False

$X = \mathbb{N}$. $\mathcal{T} = \{\emptyset, X\}$, $\mathcal{U}$ = discrete topology

$(X, \mathcal{T})$ is compact. But $(X, \mathcal{U})$ isn't since $\{\{n\} : n \in \mathbb{N}\}$ has no finite subcover

Other way around is true.

Consider any open cover $\mathcal{O}$ of $(X, \mathcal{T})$

Since $\mathcal{T} \subseteq \mathcal{U}$, $\mathcal{O}$ is also an open cover of $(X, \mathcal{U})$

Since $(X, \mathcal{U})$ is compact, $\exists$ finite subcover consisting of open sets in $\mathcal{T}$ $\Rightarrow$ $(X, \mathcal{T})$ is compact.

(8) Is $\mathbb{R}_\ell$ compact? Identify compact subsets of $\mathbb{R}_\ell$

No.

$\mathbb{R}_\ell = \bigcup_{n \in \mathbb{Z}} [n, n+1)$ $\rightarrow$ no finite subcover

Suppose $\exists$ finite subcover $[n_1, n_1+1), \ldots, [n_k, n_k+1)$
s.t $n_1 < n_2 < \ldots < n_k$, $n_i \in \mathbb{Z}$

Then $n_k+1 \notin \bigcup_{i=1}^{k} [n_i, n_i+1)$ $\rightarrow$ Contd^n

Date :

Consider compact $K \subset \mathbb{R}_\ell$

① Closed : $\mathbb{R}_\ell$ is Hausdorff, K must be closed

② Bounded : Apply covering by $(n, n+1)$ argument

③ Countable : Fix $x \in K$ & consider the following open cover of K

$\{[x, \infty)\} \cup \{(-\infty, x - \frac{1}{n}) : n \in \mathbb{Z}_{>0}\}$

Since K is compact, it has finite subcover.

Hence, $\exists$ real number a_x s.t interval $(a_x, x]$ contains no pt of K apart from x. This is true for all $x \in K$

Now choose rational no. $q_x \in (a_x, x] \cap \mathbb{Q}$.

Since the intervals $(a_x, x]$ parametrized by $x \in K$ are pairwise disjoint, the fn $q : K \to \mathbb{Q}$ is injective, so K is at most countable. $x \mapsto q_x$

Date : ______________

(10) Is $[0,1]$ compact in $\mathbb{R}$ with the topo.
$\mathcal{T} = \{A : \mathbb{R}\setminus A$ is countable or all of $\mathbb{R}\}$

Consider the sequence of pts. $\{x_n\}_{n=1}^{\infty}$ with $x_n = \frac{1}{n}$

Let $F_1 = \{x_n\}_{n=1}^{\infty}$
$F_2 = \{x_n\}_{n=2}^{\infty}$
$\vdots$

Since each F_k, $k \geq 1$ is countable, $U_k = \mathbb{R}\setminus F_k$ are open.

The union $\bigcup_{n=1}^{\infty} U_n$ covers $[0,1]$ (Since any point $\frac{1}{n}$ is contained in U_{n+1})

Suppose $[0,1]$ is compact. Then $\exists$ finite subcover $\bigcup_{i=1}^{k} U_{n_i}$

Then $N = \max_{1\leq i\leq k} n_i$

The point $\{\frac{1}{N+1}\}$ is not in the finite union $\rightarrow$ Contd^n

Hence $[0,1]$ is not Compact under co-countable topology

(11) Let X, Y be topo sp. If Y is compact, show that projection map $P_1 : X \times Y \to X$ is closed map

Consider a closed set $F \subset X\times Y$. We will show $P_1(F)$ is closed

Let $x_0 \notin P_1(F)$. $\Rightarrow (x_0, y) \notin F \quad \forall y \in Y$

$\Rightarrow \{x_0\}\times Y \subset (X\times Y)\setminus F$

Date :

Since F is closed, $(X \times Y) \setminus F$ is open.

For each $y \in Y$, $\exists$ open sets $U_y \subset X$ & $V_y \subset Y$

s.t $(x_0, y) \in U_y \times V_y \subset (X \times Y) \setminus F$

The sets $\{V_y\}_{y \in Y}$ form an open cover of Y.

Since Y is compact, $\exists$ finitely many $V_{y_1}, \dots, V_{y_n}$ that cover Y.

Let $U = \bigcap_{i=1}^{n} U_{y_i}$

U open in X. $x_0 \in U$

Claim: $U \cap p_1(F) = \emptyset$.

Take any $x \in U$. Suppose $x \in p_1(F) \Rightarrow (x, y) \in F$ for some $y \in Y$.

Since $\{V_{y_i}\}$ cover Y, $y \in V_{y_i}$ for some i

Thus $(x, y) \in U_{y_i} \times V_{y_i} \subset (X \times Y) \setminus F \Rightarrow (x, y) \notin F \rightarrow$ Contdⁿ

So, $U \cap p_1(F) = \emptyset \Rightarrow U \subset X \setminus p_1(F)$

$\Rightarrow p_1(F)$ is closed

$\Rightarrow p_1$ is a closed map.

Date :

(11) Let X, Y be topo. sp. & Y be compact Hausdorff

Then f is cts. iff the graph of f,

$G_f = \{(x, f(x)) : x \in X\}$ is closed in $X \times Y$.

($\Rightarrow$) Let $(x_0, y_0) \notin G_f$. Then $y_0 \neq f(x_0)$

Since Y is Hdf, $\exists$ disjoint open sets V, W s.t. $y_0 \in V$, $f(x_0) \in W$, $V \cap W = \emptyset$

By continuity of f, $U = f^{-1}(W)$ is open in X & $x_0 \in U$.

Consider the product nbd $U \times V$

$(x_0, y_0) \in U \times V$

If $(x, y) \in U \times V$, then, $f(x) \in W$ $(\because U = f^{-1}(W))$

Since $y \in V \Rightarrow y \neq f(x)$ $(\because V \cap W = \emptyset)$

Thus $U \times V \subset (X \times Y) \setminus G_f$.

So, complement of G_f is open $\Rightarrow G_f$ is closed.

($\Leftarrow$) We will show for any closed set $C \subset Y$,

$f^{-1}(C)$ is closed in X.

$$f^{-1}(C) = p_1(G_f \cap (X \times C)), \text{ where } p_1 : X \times Y \to X \text{ is projection}$$

G_f is closed (given)

$X \times C$ is closed (since C is closed)

So, $G_f \cap (X \times C)$ is closed in $X \times Y$

Date :

Since Y is compact, $p_1 : X \times Y \to X$ is a closed map.

Therefore, $p_1(G_f \cap (X \times C))$ is closed in X

$\Rightarrow f^{-1}(C)$ is closed in X.

Hence, f is cts.

TUTORIAL 9

Date :

(1) Show that X is normal, iff for any closed A & an open set U s.t $A \subseteq U$, $\exists$ an open set V s.t $A \subseteq V \subseteq \overline{V} \subseteq U$

($\Rightarrow$) Given X is normal, $\underset{\text{closed}}{A} \subseteq \underset{\text{open}}{U}$

Then $X \setminus U$ is closed s.t $A \cap (X \setminus U) = \emptyset$

Since X is normal, (non-empty) $\exists$ disjoint open sets V_1, V_2 s.t $A \subseteq V_1$, $(X \setminus U) \subseteq V_2$, $V_1 \cap V_2 = \emptyset$

$\Rightarrow X \setminus V_2 \subseteq U$

Since $V_1 \subseteq \underbrace{X \setminus V_2}_{\text{closed}} \subseteq U \Rightarrow V_1 \subseteq \overline{V_1} \subseteq U$

$\Rightarrow A \subseteq V_1 \subseteq \overline{V_1} \subseteq U$

($\Leftarrow$) Let $A, B \subseteq X$ be disjoint closed sets.

So, $U = X \setminus B$ is open & $A \subseteq U$.

By hypothesis, $\exists$ open set V s.t $A \subseteq V \subseteq \underbrace{\overline{V} \subseteq U}$

$\overline{V} \subseteq X \setminus B$

$\Rightarrow B \subseteq \underbrace{X \setminus \overline{V}}_{\text{open}}$ & $V \cap (X \setminus \overline{V}) = \emptyset$

& $A \subseteq V$ open

Hence, X is normal.

(2) Show that $\mathbb{R}_K$ which is $\mathbb{R}$ with basis $\{(a,b) : a,b \in \mathbb{R}\}$ $\cup \{(a,b) - K : a,b \in \mathbb{R}\}$ with $K = \{1/n : n \in \mathbb{N}\}$ is Hausdorff.

Since $\mathbb{R}$ is Hausdorff & $\mathbb{R}_K$ contains basic open sets of $\mathbb{R}$

$\Rightarrow \mathbb{R}_K$ is Hdf.

Date : ______________

(3) A topological sp. is a regular sp if a pt. & closed set can be separated by disjoint open sets & T_3 if also T_1

Show that a space X is regular if for every $a \in X$ & open set U s.t $a \in U$, $\exists$ open set $V \subseteq X$ s.t $a \in V \subseteq \bar{V} \subseteq U$

Consider a pt. $a \in X$ & closed $W \subseteq X$ s.t $a \notin W$

$\Rightarrow a \in X \setminus W$ open

By hypothesis, $\exists$ open set $V \subseteq X$ s.t $a \in V \subseteq \bar{V} \subseteq X \setminus W$

$\Rightarrow W \subseteq X \setminus \bar{V}$ open

& $V \cap (X \setminus \bar{V}) = \emptyset$

Hence, X is regular

(4) Prove that subspace of a regular space is regular

Let X be regular & $A \subseteq X$.

Consider $a \in A$, $V \subseteq A$ closed s.t $a \notin V$

Since $a \in X$ & $V = \underbrace{W}_{\text{closed in } X} \cap A \Rightarrow a \notin W$

So, $\exists$ open sets U_1, U_2 s.t $a \in U_1$, $W \subseteq U_2$, $U_1 \cap U_2 = \emptyset$

($\because X$ is regular)

$\Rightarrow V \subseteq \underbrace{U_2 \cap A}_{\text{open in } A}$

$\Rightarrow a \in \underbrace{(U_1 \cap A)}_{\text{open in } A}$

$(U_1 \cap A) \cap (U_2 \cap A) = \emptyset$

Hence, A is regular

Date :

(5) Show that product of regular space is regular

Let $\{X_i\}_{i\in\mathcal{I}}$ be a family of regular spaces, & let $X = \prod_{i\in\mathcal{I}} X_i$ with product top.

Consider $x \in X$ & closed set $F \subset X$ s.t $x \notin F$

Since F is closed & $x \notin F$, $\exists$ basic open set U s.t

$$x \in U \subset X \setminus F$$

where $U = \prod_{i\in\mathcal{I}} U_i$ with each U_i open in X_i & $U_i = X_i$ for all but finitely many indices

Let those finitely many indices be $i_1, i_2, \dots, i_n$

For each $1 \le k \le n$, $x_{i_k} \in U_{i_k}$ & X_{i_k} is regular

So, $\exists$ open set V_{i_k} s.t $x_{i_k} \in V_{i_k} \subseteq \overline{V_{i_k}} \subseteq U_{i_k}$

For all other indices let $V_i = X_i$

Now, define $V = \prod_{i\in\mathcal{I}} V_i$

Then V is open & $x \in V$ & $\overline{V} \subseteq \prod_{i\in\mathcal{I}} \overline{V_i}$

Since for each relevant coordinate $\overline{V_{i_k}} \subseteq U_{i_k} \Rightarrow \prod_{i\in\mathcal{I}} \overline{V_i} \subseteq \prod_{i\in\mathcal{I}} U_i = U$

Hence $\overline{V} \subset U \subset X \setminus F \Rightarrow F \subset \underbrace{X \setminus \overline{V}}_{\text{open}}$ & $x \in \underbrace{V}_{\text{open}}$

& $V \cap (X \setminus \overline{V}) = \emptyset$

T'fore X is regular.

Date : ____________

(6) Show that $\mathbb{R}_l \times \mathbb{R}_l$ is regular

It suffices to show $\mathbb{R}_l$ is regular ($\because$ product of regular spaces is regular)

Consider $x \in \mathbb{R}_l$ & open $U \subseteq \mathbb{R}_l$ s.t $x \in U$.

Then $\exists$ basic open set $[x, x+\epsilon)$ for $\epsilon > 0$ s.t $x \in [x, x+\epsilon) \subseteq U$

Now, $\overline{[x, x+\epsilon)} = [x, x+\epsilon)$ ($\because [a,b)$ is both open & closed in $\mathbb{R}_l$)

Hence, $\exists$ open set V s.t $x \in V \subseteq \overline{V} \subset U$

T'fore $\mathbb{R}_l$ is regular.

(7) Show $\mathbb{R}_K$ is not regular

Consider 0 & closed $K \subseteq \mathbb{R}_K$

$$\mathbb{R}_K \setminus K = \bigcup_{a<b} \underbrace{((a,b) \setminus K)}_{\text{open}} \Rightarrow K \text{ is closed in } \mathbb{R}_K$$

Suppose $\exists$ open sets $U, V \subseteq \mathbb{R}_K$ s.t $0 \in U$, $K \subseteq V$, $U \cap V = \emptyset$

Since U is open, $\exists$ basic open set $(-\epsilon, \epsilon) \setminus K \subset U$ s.t $0 \in (-\epsilon, \epsilon) \setminus K$

Sim, since V is open, for any $\frac{1}{n} \in K$, $\exists\ (c_n, d_n) \subseteq V$ s.t $\frac{1}{n} \in (c_n, d_n)$

Consider $n > \frac{1}{\epsilon} \Rightarrow \frac{1}{n} < \epsilon \Rightarrow ((-\epsilon, \epsilon) \setminus K) \cap (c_n, d_n) \neq \emptyset$

$\Rightarrow U \cap V \neq \emptyset \rightarrow$ Contd^n

Hence, $\mathbb{R}_K$ is not regular.

Date :

(8) Give example of a regular space which is not normal

$\mathbb{R}_\ell \times \mathbb{R}_\ell$ is not normal

Consider $\Delta = \{(x, -x) : x \in \mathbb{R}\}$
$K = \{(x, -x) : x \in \mathbb{Q}\}$
$L = \{(x, -x) : x \in \mathbb{R} \setminus \mathbb{Q}\}$

① Δ is closed (show $\mathbb{R}_\ell \times \mathbb{R}_\ell \setminus \Delta$ is open)
② Δ is discrete ($([x, x+\epsilon) \times [-x, -x+\epsilon)$ only intersects Δ at $(x, -x)$)
in subsp top.

③ Subsets of Δ are closed in Δ ($\because$ every subset of discrete sp is both open & closed)

Hence, K & L are closed in Δ
$\Rightarrow$ K & L are closed in $\mathbb{R}_\ell \times \mathbb{R}_\ell$.

Now suppose $\exists$ open sets $K \subseteq U$, $L \subseteq V$, $U \cap V = \emptyset$

(K is dense in Δ & L is dense in Δ. We can make U, V intersect)
Take $(x, -x) \in L$ (i.e $x \in \mathbb{R} \setminus \mathbb{Q}$).
Then $\exists$ $[a, b) \times [c, d) \subseteq V$ containing $(x, -x)$
We will show $\exists$ pt. $(q, -q) \in [a, b) \times [c, d)$, $q \in \mathbb{Q}$

Let $I = [a, b) \cap (-d, -c]$. Since $x \in I$, I is non-empty.
Since $x \in [a, b)$ & $x \in (-d, -c]$, $\exists$ open interval around x contained in I.
By density of rationals, it also contains a pt. q. $\Rightarrow (q, -q) \in [a, b) \times [c, d) \Rightarrow K \cap V \neq \emptyset$
Contd^n $\Rightarrow U \cap V \neq \emptyset$

Date :

(9) Show that if Y is Hausdorff & $f, g : X \to Y$ are cts. fn's, then $\{x \in X : f(x) = g(x)\}$ is a closed set.
Prove then that if two cts. fn's into Y agree on a dense set, then they are equal.

Consider the map $F : X \to Y \times Y$
$$x \mapsto (f(x), g(x))$$

Since f, g are cts., F is cts.
Consider the diagonal, $\Delta = \{(y, y) \in Y \times Y : y \in Y\}$

If Y is Hausdorff, Δ is closed in $Y \times Y$.
This is because if $(y_1, y_2) \notin \Delta \Rightarrow y_1 \neq y_2$
$\Rightarrow \exists$ open sets. $y_1 \in U$, $y_2 \in V$, $U \cap V = \emptyset$
$\Rightarrow (y_1, y_2) \in U \times V$ is an open set containing (y_1, y_2) disjoint from Δ.
$\Rightarrow$ Complement of Δ is open
$\Rightarrow$ Δ is closed

Let $E = \{x \in X : f(x) = g(x)\}$
$= F^{-1}(\Delta)$
Since F is cts. & Δ is closed $\Rightarrow$ E is closed.

Date :

Let $D \subset X$ be a dense set & $f(x) = g(x)\ \forall x \in D$

Since E is closed $\Rightarrow \bar{D} \subset E$

But D is dense, so $\bar{D} = X \Rightarrow E = X$

So, $f(x) = g(x)\ \forall x \in X$

(10) <u>Show that subsp. of a completely regular space is completely regular</u>

Let X be a completely regular & $Y \subseteq X$ with subsp topo.

Consider $y \in Y$ & $F \subset Y$ closed s.t $y \notin F$

Since F is closed in Y, $\exists$ closed set $C \subset X$ s.t $F = C \cap Y$.

Also, $y \notin C$

Since X is completely regular, $\exists$ cts. fxn $f : X \to [0,1]$
$f(y) = 0$, $f(C) = \{1\}$

Define $g = f_Y : Y \to [0,1]$
$g(y) = 0$

For any $z \in F$, since $F \subset C$, $g(z) = f(z) = 1$
$\Rightarrow g(F) = \{1\}$

Hence Y is completely regular.

Date : ____________

(11) Show that product of completely regular space is completely regular.

Let $\{X_i\}_{i\in J}$ be completely regular spaces. & let $X = \prod_{i\in J} X_i$ with product topology.

A space is completely regular iff $\forall x \in X$ & every open nbd $x \in U$, $\exists$ cts $f: X \to [0,1]$ s.t $f(x) = 0$, $f \equiv 1$ on $X \setminus U$.

Let $x = (x_i) \in X$. & let U be an open set with $x \in U$.
Then $\exists$ basic open set $B = \prod_{i\in J} U_i$ s.t $x \in B \subseteq U$
where $U_i \subseteq X_i$ is open & $U_i = X_i$ for all but finitely many indices $i_1, \dots, i_n$

For each $k = 1, \dots, n$, since X_{i_k} is completely regular,
& $x_{i_k} \in U_{i_k}$, $\exists$ cts. f^n $f_{i_k}: X_{i_k} \to [0,1]$ s.t
$f_{i_k}(x_{i_k}) = 0$ & $f_{i_k} \equiv 1$ on $X_{i_k} \setminus U_{i_k}$

Let $\pi_{i_k}: X \to X_{i_k}$ be the proj. maps (cts.)
Def $g_k = f_{i_k} \circ \pi_{i_k} : X \to [0,1]$

Def. $f(x) = \max\{g_1(x), \dots, g_n(x)\}$
f is cts (finite max of cts f^ns)
$f(x) = 0$ since $g_k(x) = 0$

Date :

If $y \notin B$, then for some k, $y_k \notin U_k$

So, $f_k(y_k) = 1 \Rightarrow g_k(y) = 1$

$\Rightarrow f(y) = 1$

$\Rightarrow f \equiv 1$ on $X \setminus B \supset X \setminus U$

Hence, $\exists$ cts fx^n $f : X \to [0,1]$ s.t $f(x) = 0$ & $f \equiv 1$ on $X \setminus U$.
T'fore X is completely regular.

(12) Show that $\mathbb{R}_\ell \times \mathbb{R}_\ell$ is completely regular

It suffices to show $\mathbb{R}_\ell$ is completely regular

Consider $x \in \mathbb{R}_\ell$ & closed $F \subseteq \mathbb{R}_\ell$ s.t $x \notin F$.
Since $\mathbb{R}_\ell \setminus F$ is open, $\exists$ basic open set $x \in [x, x+\epsilon) \subseteq \mathbb{R}_\ell \setminus F$

Define $f(y) = 1 - \chi_{[x, x+\epsilon)} = \begin{cases} 0, & y \in [x, x+\epsilon) \\ 1, & y \notin [x, x+\epsilon) \end{cases}$

f is cts, as preimages of closed sets $\{0\}, \{1\}$
are closed sets.

Then $f : X \to [0,1]$, $f(x) = 0$, $f(F) = 1$.
Hence, $\mathbb{R}_\ell$ is completely regular.

TUTORIAL 10

Date :

(1) Prove that product of Tychonoff space is Tychonoff

Let $\{X_\alpha\}_{\alpha \in A}$ be a family of Tychonoff spaces
By assumption, for each α, $\exists$ embedding $\varphi_\alpha : X_\alpha \hookrightarrow [0,1]^{I_\alpha}$

Consider the product space $X = \prod_{\alpha \in A} X_\alpha$
Define $\Phi : \prod_{\alpha \in A} X_\alpha \to \prod_{\alpha \in A} [0,1]^{I_\alpha}$

by $\Phi((x_\alpha)) = ((\varphi_\alpha(x_\alpha))$
which is just the product of embeddings

Now, $\prod_{\alpha \in A} [0,1]^{I_\alpha} \simeq [0,1]^{\bigsqcup_{\alpha \in A} I_\alpha}$

Each φ_α is cts. $\Rightarrow$ Φ is cts.
Each φ_α is inj. $\Rightarrow$ Φ is inj.
Product topology ensures Φ is an embedding (product of embeddings in an embedding)

$$\prod_\alpha X_\alpha \hookrightarrow [0,1]^{\bigsqcup_\alpha I_\alpha}$$

Since product embeds into a cube, it is Tychonoff

(2) Show that Tietze extention then implies Urysohn's lemma & hence normality
(Already done in notes)

Date : ______________

(3) Show that a connected T_4 space with more than one point is uncountable

Let X be a connected T_4 space with at least 2 distinct p
We prove X is uncountable.

Take two distinct pts. $x \neq y \in X$. Since X is Hausdorff,
$\{x\}$ & $\{y\}$ are closed. Because X is normal, by
Urysohn's lemma, $\exists$ cts. fn $f : X \to [0,1]$ s.t

$f(x) = 0$, $f(y) = 1$

Since X is conn. & f is cts, the image $f(X)$ is a conn. subset of $[0,1]$.
The only conn. subsets of $[0,1]$ are intervals, so $f(X)$ is an interval containing both 0 & 1.
Hence $f(X) = [0,1]$

Thus $\exists$ surjection $f : X \to [0,1]$.
Since $[0,1]$ is uncountable, any set that maps onto it must also be uncountable.

ote: The result also holds for a countable, regular (T_3) space. (Lindelöf)

Date :

(4) Show that if X is Hausdorff, then the following statements are equal. A Hausdorff space is locally compact if it satisfies either of these statements

a) For every $x \in X$, $\exists$ compact set $W \subseteq X$ s.t it contains an open set V with $x \in V \subseteq W$
(This is saying every set has a cmpt. nbd)

b) For every $x \in X$ & open set $x \in U \subseteq X$, $\exists$ open set $V \subseteq X$ s.t if $x \in V \subseteq \overline{V} \subseteq U$ & $\overline{V}$ is cmpt.
(This is saying that every open set contains a cmpt. nbd)

$b \Rightarrow a$

For every $x \in X$ & open $U \ni x$, $\exists$ open V s.t $x \in V \subseteq \underset{\text{cmpt.}}{\overline{V}} \subseteq U$

fix $x \in X$. Take $U = X$. (open).
Then $\exists$ open set V s.t $x \in V \subseteq \underset{\text{cmpt.}}{\overline{V}} \subseteq X$

Thus V is an open nbd of x contained in the compact set $W = \overline{V}$.

$a \Rightarrow b$

For each $x \in X$, $\exists$ cmpt. W & open V s.t $x \in V \subseteq W$.
Now, let $x \in X$ & U be any open set $x \in U$
We must find an open V s.t $x \in V \subseteq \underset{\text{cmpt}}{\overline{V}} \subseteq U$

Date :

(5) When is a compact sp. (not necessarily Hdf) locally cmpt if (a) is taken as def^n or (b) is or both or neither?

Every compact space satisfies (a). So it doesn't distinguish anything

In a compact sp., (b) does not hold in general
Hence (b) is the correct notion.

(6) Prove or disprove
(a) $\mathbb{Q}$ is locally compact
(b) $\mathbb{R}_\ell$ is locally compact
(c) Let $f: X \to Y$ be cts map of Hdf sp.
Then X is loc. cmpt. $\Rightarrow f(X)$ is loc. cmpt.

(a) False
By def^n (b) in above question, every pt. must have a nbd whose closure is cmpt.

So, we will show, for any $q \in \mathbb{Q}$ & any nbd U of q in $\mathbb{Q}$, the closure $\overline{U}$ (in $\mathbb{Q}$) is not cmpt.

Any open nbd of $q \in \mathbb{Q}$ has the form $U = (a,b) \cap \mathbb{Q}$ for $a, b \in \mathbb{R}$ s.t $a < q < b$
Closure of U in $\mathbb{Q}$ is $Cl_{\mathbb{Q}}(U) = [a,b] \cap \mathbb{Q}$

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Use (a) inside U: $\exists$ open V_0 & cmpt. W s.t $x \in V_0 \subseteq W$

Replace V_0 by $V_0 \cap U$ which open & contains x

Thus $x \in V_0 \cap U \subseteq W \cap U$

So, we may assume $x \in V_0 \subseteq W \cap U$

Since W is cmpt. & X is Hdf,

- W is closed
- $W \setminus U$ is closed in X, hence cmpt.

Also $x \notin W \setminus U$.

☆ If K is cmpt. & $x \notin K$, $\exists$ disjoint open sets separating x & K

So, we can separate, hence $\exists$ open V, O s.t

$x \in V$, $W \setminus U \subseteq O$, $V \cap O = \emptyset$.

(we can ensure $\bar{V} \subseteq W$ by taking V as $V \cap V_0$

Since V is disjoint from O & $W \setminus U \subseteq O$, we get

$\bar{V} \cap (W \setminus U) = \emptyset$

Thus, $\bar{V} \subseteq W \cap U \subseteq U$

We have $\bar{V} \subseteq W$ & W is cmpt. $\Rightarrow$ $\bar{V}$ is cmpt.

closed in X

$\Rightarrow$ closed in W

Hence, we have constructed open set V s.t $x \in V \subseteq \underset{\text{cmpt}}{\bar{V}} \subseteq U$

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We show $Cl_{\mathbb{Q}}(U)$ isn't cmpt.

Pick an irrational no. $\alpha \in (a,b)$

$\exists$ seq. $(q_n) \subset \mathbb{Q}$ s.t $q_n \to \alpha$

Each $q_n \in [a,b] \cap \mathbb{Q}$ but $\alpha \notin \mathbb{Q}$.

So, the sequence has no limit in $\mathbb{Q}$. Hence no conv. subseq.

Thus $[a,b] \cap \mathbb{Q}$ is not sequentially cmpt., hence not cmpt

Hence, $\mathbb{Q}$ is not locally compact.

b) False

Suppose it is. Then every x has a cmpt. nbd K s.t

$x \in [a,b) \subseteq K \subseteq \mathbb{R}$. But for a suitable n,

$$(-\infty, a) \cup \left(\bigcup_{j=n}^{\infty} [a, b - \tfrac{1}{j}) \right) \cup (b, \infty)$$

is an open cover of K with no finite subcover

Hence $\mathbb{R}_l$ is not loc. cmpt.

c) False

$$id : \underbrace{(\mathbb{R}, \text{discrete})}_{X} \to \underbrace{\mathbb{R}_l}_{Y}$$

cts.

X is loc. cmpt. $\{x\}$ is its own cmpt. nbd.

but $f(X)$ is not

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(7) Let $(X, \mathcal{T})$ be a loc. cmpt Hdf sp.
Let $X^* = X \sqcup \{+\}$. Define $\mathcal{U} = \mathcal{T} \sqcup \{\{+\} \cup X \setminus K : K \text{ cmpt in } X\}$

a) Show that this defines a topo. on X^*
b) Show that X^* is a cmpt Hdf sp
c) If $X = \mathbb{R}$, show that X^* is homeomp. to S^1.

a) ① $\emptyset \in \mathcal{T} \subset \mathcal{U}$

$$X^* = X \cup \{+\} = \{+\} \cup X \setminus \underbrace{\emptyset}_{\text{cmpt.}} \Rightarrow X^* \in \mathcal{U}.$$

② Arbitrary unions

Let $\{U_i\}_{i \in \mathcal{I}} \subset \mathcal{U}$

2.1 None contain $+$

Each $U_i \in \mathcal{T} \Rightarrow \bigcup_{i \in \mathcal{I}} U_i \in \mathcal{T} \subset \mathcal{U}$

2.2 At least one contains $+$

Each such set has form $U_i = \{+\} \cup (X \setminus K_i)$

Then $\bigcup U_i = \{+\} \cup (X \setminus K_i) = \{+\} \cup X \setminus (\underbrace{\cap K_i}) \in \mathcal{U}$

K_i s cmpt.

$\cap K_i$ closed in cmpt set $\Rightarrow$ Cmpt.

③ Finite intersections

Take $U_1, U_2 \in \mathcal{U}$

3.1 Both in $\mathcal{T}$. $U_1 \cap U_2 \in \mathcal{T} \subseteq \mathcal{U}$

3.2 One contains $+$: $U_1 = \{+\} \cup (X \setminus K)$, $U_2 \in \mathcal{T} \Rightarrow U_1 \cap U_2 = U_2 \cap \underbrace{(X \setminus K)}_{\text{open}} \in \mathcal{U}$

3.3 Both contain $+$: $U_1 = \{+\} \cup (X \setminus K)$, $U_2 = \{+\} \cup (X \setminus K)$ $\Rightarrow U_1 \cap U_2 = \{+\} \cap (X \setminus (\underbrace{K_1 \cup K_2})) \in \mathcal{U}$

finite union of cmpt sets is cmpt.

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(b) Done at the back of notebook

(c) Define the map

$$f : \mathbb{R}^* \to S^1 \subset \mathbb{R}^2$$

$$f(x) = \left(\frac{2x}{1+x^2} , \frac{x^2-1}{1+x^2} \right), \ x \in \mathbb{R}$$

$$\& \ f(+) = (0,1)$$

__Bij:__ For $x \in \mathbb{R}$, this parametrizes all of S^1 except $(0,1)$.

The pt. $+$ maps to $(0,1)$

So, f is bij.

__Continuity:__ Continuous on $\mathbb{R}$.

At $+$: nbds of $+$ are $\{+\} \cup (\mathbb{R} \setminus K)$ where K is cmpt.

So approaching $+$ means $x \to \pm\infty$

Since $\lim_{x \to \pm\infty} f(x) = (0,1)$, f is cts. at $+$.

__Homeomp:__ $\mathbb{R}^*$ is cmpt. & S^1 is Hdf.

A cts. bij. from cmpt to Hdf is homeomp.

So, $\mathbb{R}^* \simeq S^1$

(7) b) Only need to show $\{+\}$ & any pt. in X can be separated.

Consider $a \in X$. $\exists$ cmpt. W s.t. $a \in \underset{\text{open}}{V} \subseteq W$

Then $\{+\} \subseteq \underbrace{\{+\} \cup X \setminus W}_{\text{open in } X^*}$ & $a \in \underbrace{V}_{\text{open in } X^*}$ & $(\{+\} \cup X \setminus W) \cap V = \emptyset$

Now, we'll show $\{+\} \cup X$ is compact.
Let $\{A_\alpha\}_{\alpha \in \Delta}$ be an open cover of $\{+\} \cup X$ (say Y)

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$\because$ $t \in Y \Rightarrow \exists A_{\alpha_1} \in \mathcal{U}$ s.t $t \in A_{\alpha_1} = Y - K$ for some compact set in X.

$\Rightarrow K \subseteq (\cup A_\alpha)$

$\Rightarrow K \subseteq \underbrace{(\cup A_\alpha) - \{t\}}$

are all open in X

Since K cmpt in X $\Rightarrow$ K is closed ($\because$ X is Hausdorff)

$\Rightarrow$ K has finite subcover $A_{\alpha_2}, \dots, A_{\alpha_k}$

$\Rightarrow Y = A_{\alpha_1} \cup \dots \cup A_{\alpha_k}$

$\Rightarrow Y$ is cmpt

<u>Note</u>: Subspace topology of $(X^*, \mathcal{U})$ on X is $\mathcal{T}$

TUTORIAL 11

Date :

PART I

A locally compact Hausdorff space is Tychonoff

(1) X is loc. cmpt. Hdf $\Rightarrow \exists Y \supseteq X$ s.t Y is cmpt. Hdf

$\Rightarrow X$ is subsp. of cmpt. Hdf

$\Rightarrow X$ is Tychonoff.

If X is second countable, every open cover of X has a countable subcover

(2) Consider open cover $\mathcal{O}$. (i.e X is Lindelöff)

For any $a \in X$, $\exists\, U \in \mathcal{O}$. Then $\exists\, B_j$ in base s.t $a \in B_j \subseteq U_a$

Consider the coll. $S_j = \{U \subseteq \mathcal{O} : B_j \subseteq U\}$ by varying over $a \in X$.

Choosing any U_j from each S_j gives us the required countable subcover

If a space is T_3 & Lindelöff, then it is T_4

(3) Consider A, B disj. closed sets.

$\because X$ is T_3, $\forall\, a \in A$, $\exists$ open $a \in U_a$ & $B \subseteq V_a$ s.t $U_a \cap V_a = \emptyset$

Also $A = \bigcup_{a \in A} U_a$ has a countable subcover $A \subseteq \bigcup_{i=1}^{\infty} U_i$

Sim. $B = \bigcup_{a \in A} V_a$, so $B \subseteq \bigcup_{i=1}^{\infty} V_i$

Consider $W_n = \bigcup_{i=1}^{n} U_i - \bigcup_{i=1}^{n-1} \overline{V_i}$, $T_n = \bigcup_{i=1}^{n} V_i - \bigcup_{i=1}^{n-1} \overline{U_i}$

Check $A \subseteq \bigcup_{n=1}^{\infty} W_n$ & $B \subseteq \bigcup_{n=1}^{\infty} T_n$, $T_n \cap W_n = \emptyset$

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~~Claim: $A \cap \bigcup_{i=1}^{n} U_i = A \cap W_n$~~

~~$a \in A$, $a \in \bigcup_{j=1}^{n} U_j \Rightarrow a \in U_j$ for some j~~

~~$\Rightarrow a \notin V_j$~~

~~$\Rightarrow a \notin \overline{V_j}$?~~

$\forall a \in A$, $\exists$ open U_a, U_a' s.t $a \in U_a$, $B \subseteq U_a'$ & $U_a \cap U_a' = \emptyset$

Then $A \subseteq \bigcup_{a \in A} U_a$

$U_a \subseteq X - U_a' \subseteq X - B$

$\Rightarrow \overline{U_a} \subseteq X - B$

Choose $\forall b \in B$, $\exists$ open V_b, V_b' s.t $b \in V_b$, $A \subseteq V_b'$ & $V_b \cap V_b' = \emptyset$

Then $B \subseteq \bigcup_{b \in B} V_b$ $\Rightarrow \overline{V_b} \subseteq X - A$

Use Lindeloff to get countable subcover of A & B

$A \subseteq \bigcup_{i=1}^{\infty} U_i$, $B \subseteq \bigcup_{i=1}^{\infty} V_i$

Def. $W_1 = U_1 - \overline{V_1}$, $L_1 = V_1 - \overline{U_1}$

$W_2 = U_2 - \overline{V_1 \cup V_2}$, $L_2 = V_2 - \overline{U_1 \cup U_2}$

$\vdots$ $\vdots$

$W_n = U_n - \overline{V_1 \cup \ldots \cup V_n}$, $L_n = V_n - \overline{U_1 \cup \ldots \cup U_n}$

Show $\left(\bigcup_{i \in \mathbb{Z}_{>0}} W_i \right) \cap \left(\bigcup_{i \in \mathbb{Z}_{>0}} L_i \right) = \emptyset$

Also, $A \subseteq \bigcup_{i \in \mathbb{Z}_{>0}} W_i$, $B \subseteq \bigcup_{i \in \mathbb{Z}_{>0}} L_i$

open open

Every separable metric space is second countable Date :

(4) Let (X,d) be separable. Then $\exists$ a countable dense set $D = \{x_1, x_2, \dots\}$

Let $Q^+ = \{q \in Q : q > 0\}$ which is also countable

Define $\mathcal{B} = \{B(x,q) : x \in D,\ q \in Q^+\}$

$\mathcal{B}$ is countable

$\mathcal{B}$ is a basis : For every open set $U \subseteq X$ & every $x \in U$,

$\exists\, B(d,q) \in \mathcal{B}$ s.t $x \in B(d,q) \subseteq U$

Let $x \in U$. Since U is open, $\exists\, \epsilon > 0$ s.t $B(x,\epsilon) \subseteq U$

Since D is dense, $\exists\, d \in D$ s.t $d(x,d) < \epsilon/2$. Now choose $q \in Q^+$ s.t $d(x,d) < q < \epsilon/2$

Claim : $B(d,q) \subseteq B(x,\epsilon)$ since for any $y \in B(d,q)$

$d(x,y) \leq d(x,d) + d(d,y) < \frac{\epsilon}{2} + q < \epsilon$

(5) Consider any sub-basic set $\prod U_i$, s.t $U_i = (a_i, b_i)\ \forall\, i = i_1, \dots, i_n$ & $U_i = \mathbb{R}$ for rest i

Let $X = \{(x) \in Q^{\mathbb{N}} : \exists M > 0 \text{ s.t } x_i = 0\ \forall\, i \geq M\}$

Then $X \subseteq Q^{\mathbb{N}}$ (show X is countable)

& that X is dense (intersects every basic open set)

Show $\mathbb{R}^{\mathbb{N}}$ is separable

Let X be T_4 & second countable. Show that there is an embedding

Date : from $X \to I^{|\mathbb{N}|}$ (use Urysohn's fn's to separate the open sets in base)

(6) We need to collect cts. fn's $X \to I$ s.t it separates pts. from closed sets. Rest follows from them.

Since X is T_4, Urysohn's lemma gives us the functions. Consider all pairs i, j s.t $\overline{B_i} \subset B_j \Rightarrow \overline{B_i}$ & $X \setminus B_j$ are closed sets.
(countable)

$f_{ij} : X \to [0,1]$, $f_{ij}(\overline{B_i}) = 0$, $f_{ij}(X \setminus B_j) = 1$

(7) Use the above statements to prove the following are equivalent for a T_1 topological space

a) X is regular & second countable

b) X is separable & metrizable

c) X can be embedded as a subsp of Hilbert cube $I^{|\mathbb{N}|}$

$a \Rightarrow c$

Let X be regular & second countable

X is T_1 + regular $\Rightarrow$ X is T_3

By (2), second countable $\Rightarrow$ Lindeloff

By (3), T_3 + Lindeloff $\Rightarrow$ T_4

By (6), T_4 & second countable $\Rightarrow$ embedding $X \hookrightarrow I^{|\mathbb{N}|}$ (c)

$c \Rightarrow b$

Let X embed into $I^{|\mathbb{N}|}$

$I^{|\mathbb{N}|}$ is metrizable & separable

Any subsp. of a metrizable sp. is metrizable

Any subsp. of a separable sp. is separable

T'fore, X is separable & metrizable (b)

$b \Rightarrow a$

Let X be separable & metrizable

Every metric space is regular (even T_4)

By (4), separable metric sp $\Rightarrow$ second countable

(a)

Part 2

Locally finite open covers for $\mathbb{R}$

(1) $\mathbb{R} = \bigcup_{n\in\mathbb{Z}} (n, n+2)$ → any pt is contained in at most 2 sets

To prove $\mathbb{R}$ is paracompact, take any open cover $\mathcal{O}$ & intersect it with sets of the form $(n, n+2)$

(3) Every regular Lindeloff space is paracompact

Let X be a regular Lindeloff space

To show: X is paracompact

Let $\mathcal{U}$ be an open cover of X.

$\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$

$$X = \bigcup_{\alpha\in\Lambda} U_\alpha$$

$\forall x \in X$, $x \in U_\alpha$ for some α.

X is regular $\Rightarrow \exists V_\alpha$ open in X s.t. $x \in V_\alpha \subseteq \overline{V_\alpha} \subseteq U_\alpha$

$\Rightarrow X = \bigcup_{\alpha\in\Lambda} V_\alpha$ & $V_\alpha \subseteq \overline{V_\alpha} \subseteq U_\alpha$ & $X = \bigcup_{\alpha\in\Lambda} U_\alpha$

X is Lindeloff $\Rightarrow X = \bigcup_{i\in\mathbb{Z}_{>0}} V_i$ for some $V_i \subseteq \{V_\alpha : \alpha\in\Lambda\}$

$V_i \subseteq U_i \Rightarrow X = \bigcup_{i\in\mathbb{Z}_{>0}} U_i$

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Let $W_1 = U_1$

$W_2 = U_2 - \overline{V_1}$

$\vdots$

$W_n = U_n - \bigcup_{i=1}^{n-1} \overline{V_i}$

Claim: $\{W_n : n \in \mathbb{Z}_{>0}\}$ gives a locally finite refinement of $\mathcal{U}$ covering X.

By ind^n $\bigcup_{n=1}^{\infty} W_n = \bigcup_{n=1}^{\infty} U_n$

So, it is an open cover.

Let k be smallest s.t $x \in V_k$ $\Rightarrow x \notin W_i \ \forall i \geq k+1$

$\Rightarrow x \in W_i$ for some $1 \leq i \leq k$

Hence it is locally finite.

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(4) Every Hausdorff paracompact space is normal

Given X is Hausdorff paracompact.

To show : X is normal

Let A, B be closed disjoint subsets of X.

Find open set U s.t $A \subseteq U \subseteq \overline{U} \subseteq X \setminus B$

First we'll show X is regular.

A is closed. $b \notin A$.

To show: $\exists$ open set around A s.t $b \notin$ its closure.

X is Hausdorff $\Rightarrow \forall a \in A$, $\exists$ open set $a \in U_a$ & $b \notin U_a$

$$\Rightarrow A \subseteq \bigcup_{a \in A} U_a$$

$$\Rightarrow X = \bigcup_{a \in A} U_a \cup (X - A)$$

Since X is paracompact, $\exists \mathcal{W}$ an open locally finite refinement of $\{U_a : a \in A\} \cup \{X \setminus A\}$

For any $W \in \mathcal{W}$, $W \in U_a$ for some a or $W \subseteq X \setminus A$

T'fore, if $W \cap A \neq \emptyset$, then $b \notin W$.

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$\mathcal{V} = \{W : W \cap A \neq \emptyset\}$

$\Rightarrow A \subseteq \bigcup_{W \in \mathcal{V}} W$ & $b \notin \overline{W} \;\forall\, W \in \mathcal{V}$

Since $\mathcal{V}$ is locally finite, $\overline{\bigcup_{W \in \mathcal{V}} W} = \bigcup_{W \in \mathcal{V}} \overline{W}$ (Ex).

So, $A \subseteq \bigcup_{W \in \mathcal{V}} W$ & $b \in \overline{\bigcup_{W \in \mathcal{V}} W}$

$\Rightarrow X$ is regular

Use similar idea to show X is normal

(5) $\mathbb{R}_\ell$ is paracompact by $\mathbb{R}_\ell \times \mathbb{R}_\ell$ is not.

$\mathbb{R}_\ell$ is regular (For any open set U, $x \in \underbrace{[x, x+\epsilon)}_{\text{open } V = \overline{V}} \subseteq U$)

+ Lindeloff

$\Rightarrow$ paracompact

$\mathbb{R}_\ell \times \mathbb{R}_\ell$ is not normal $\Rightarrow$ Not $\underbrace{\text{Hdf}}_{\text{already Hdf.}}$ paracompact

$\Rightarrow$ Not paracompact

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Ex: $\overline{\bigcup_{W \in \mathcal{V}} W} = \bigcup_{W \in \mathcal{V}} \overline{W}$

If $x \in \overline{\bigcup_{W \in \mathcal{V}} W}$, then $\forall$ open sets $x \in U$, $U \cap \overline{\bigcup_{W \in \mathcal{V}} W}$

Around every $x \in X$, $\exists$ a nbd U which intersects only finitely many W_i. $\Rightarrow x \in \overline{W_1} \cup \overline{W_2} \cup \ldots \cup \overline{W_n}$

$\Rightarrow x \in \bigcup_{W \in \mathcal{V}} \overline{W}$

Is cts image of a paracompact sp. paracompact? No!

(?) $\mathbb{R} \times \mathbb{R} \to \mathbb{R}_\ell \times \mathbb{R}_\ell$ identity map.

(discrete) Always continuous.

(paracmpt.) (not paracmpt.)

Q. $\mathbb{R}^n$ is paracmpt.

Pf: $\mathbb{R}^n$ is regular & Lindeloff.

Date :

(6) Every discrete space is paracompact

Let X be discrete & $\mathcal{U}$ be any open cover of X.

For each $x \in X$, $\exists\ U_x \in \mathcal{U}$ s.t $x \in U_x$

Let $\mathcal{V} = \{\{x\} : x \in X\}$

$\mathcal{V}$ is a locally finite open refinement of $\mathcal{U}$.

Hence, X is paracompact