MA410

Multivariable Calculus

Instructor : Prof. Shripad Garge

Attendance policy : 100%

References : Spivak Calculus on Manifolds

Tutorial : Tuesdays

Quiz : Jan 29, Mar 26

Grading policy :

Quiz 1 - 10%

Midsem - 30%

Quiz 2 - 10%

Endsem - 50%

Main aims of Calculus: Diff. & Integration
and study rel$^n$ b/w these two.

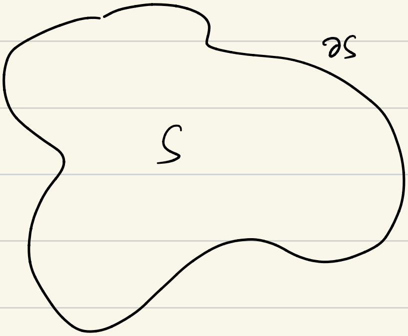
Distance, Norms

Continuity

Differentiability

Riemann integration

FTC



Fundamental Thm
of Advanced Calculus
(Stokes Thm)

Note that $\mathbb{R}^n$ is a vector field over $\mathbb{R}$.
Elements of $\mathbb{R}^n$ are called vectors.

$$\mathbb{R}^n = \{(x_1, \dots, x_n) : x_i \in \mathbb{R}\}$$

$(x_1, \cdots, x_n) + (y_1, \cdots, y_n) = (x_1+y_1, \cdots, x_n+y_n)$

$\alpha(x_1, \cdots, x_n) = (\alpha x_1, \ldots, \alpha x_n)$

$\alpha(v+w) = \alpha v + \alpha w$

<u>Norm</u>: A norm on $\mathbb{R}^n$ is a $fn^n$

$\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$ s.t

1. $\|v\| \geq 0 \quad \forall\ x \in \mathbb{R}^n$ & $\|v\| = 0$ iff $v = 0$

2. Triangle ineq. : $\|v+w\| \leq \|v\| + \|w\| \quad \forall\ v, w \in \mathbb{R}$

3. $\|\alpha w\| = |\alpha| \, \|w\| \quad \forall\ \alpha \in \mathbb{R},\ v \in \mathbb{R}^n$

eg: 1. Euclidean norm : $\|(x_1, \ldots, x_n)\| = (x_1^2 + \ldots + x_n^2)^{1/2}$

2. Sup norm : $\|(x_1, \cdots, x_n)\| = \max\{|x_i|\}_{i=1}^{n}$
(or $\infty$-norm)

3. $\|(x_1, \cdots, x_n)\|_p = (|x_1|^p + \ldots + |x_n|^p)^{1/p} \quad \forall \ 1 \leq p \leq \infty$

4. What happens if we don't take the pth power or the pth root?

Cond$^n$ 3. is violated.

A norm fx$^n$ on $\mathbb{R}^n$ gives rise to a dist. fx$^n$ on $\mathbb{R}^n$

A metric (or dist. fx$^n$) on $\mathbb{R}^n$ is a fx$^n$

$d(\cdot,\cdot) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ s.t

1. $d(v, w) \geq 0 \quad \forall \ v, w \in \mathbb{R}^n$

& $d(v, w) = 0$ iff $v = w$

2. $d(u, v) + d(v, w) \geq d(u, w) \quad \forall \ u, v, w \in \mathbb{R}^n$

3. $d(v, w) = d(w, v) \quad \forall \ v, w \in \mathbb{R}^n$

eg:

1. $d(v,w) = \|v-w\|$ for a norm $\|.\|$ on $\mathbb{R}^n$

2. Discrete metric:

$$d(v,w) = \begin{cases} 0, & \text{if } v=w \\ 1, & \text{else} \end{cases}$$

Does this metric come from a norm?

No.

open ball: If $d$ is a metric on $\mathbb{R}^n$, then an open ball of radius $r$ at the pt. $a \in \mathbb{R}^n$ is the set

$$B(a,r) := \{v \in \mathbb{R}^n : d(a,v) < r\}$$

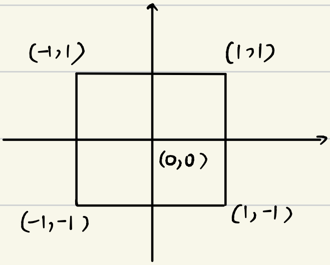
or $B_a(r)$

eg: 1. $d$: Euclidean metric on $\mathbb{R}^2$.

Unit open ball at $(0,0)$ is unit circle.

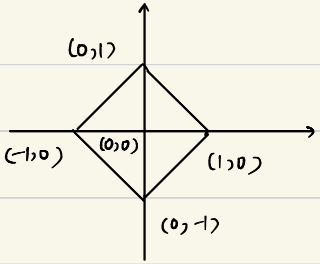
2. $d$: sup metric on $\mathbb{R}^2$

$B(0,1)$ is unit sq.



3. d: p-metric on $\mathbb{R}^2$ with $p=1$.

$B(0,1)$ is



<u>Open set in $\mathbb{R}^n$ wrt a particular metric</u>

A subset $U$ of $\mathbb{R}^n$ is said to be open wrt a metric d on $\mathbb{R}^n$, if $\forall\ a \in U$, $\exists\ r>0$ s.t $B_a(r) \subseteq U$

eg: 1. Consider $\mathbb{R}$ wrt usual dist. $d(u,v) = |u-v|$.

Here $U = (-1,1)$ is an open set.

Infact, $U = B_0(1)$.

2. $\mathbb{R}$ wrt usual metric.

$U = (-3,-1) \cup (-1,1)$ is an open set.

3. Infact, every open subset of $\mathbb{R}$, in the usual metric is at most a countable union of open intervals.

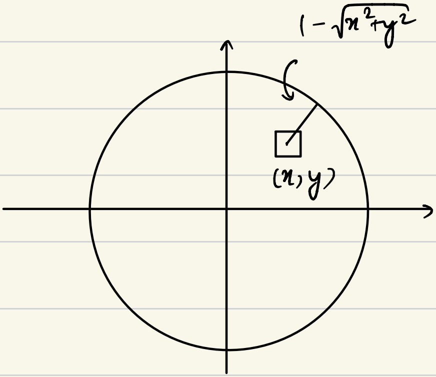
4. In $\mathbb{R}^2$, the set $(-1,1)\times(-1,1)$ is open in the Euclidean metric.

5. The set $U=\{u\in\mathbb{R}^2 : \|u\|_2<1\}$ is open in $\mathbb{R}^2$ wrt sup norm

If $u=(x,y)\in\mathbb{R}^2$, then $u\in U$ iff $x^2+y^2<1$

Choose $r<\dfrac{1-\sqrt{x^2+y^2}}{\sqrt{2}}$,

then $B_u(r)\subseteq U$



6. The set $[0,1]$ is not open in $\mathbb{R}$ in the usual metric.

For the pt. $0\in[0,1]$, $\nexists\, r$ s.t $B_0(r)=(-r,r)\subseteq[0,1]$

Negation:

Statement A: A property X is true $\forall\, u\in U$

Negation of statement A: $\exists\, u_0\in U$ s.t property X does not hold for $u_0$

Assume a metric $d$ is defined on $\mathbb{R}^n$.

A set $V$ of $\mathbb{R}^n$ is called to be closed set wrt $d$ if $\mathbb{R}^n \setminus V$ is open.

Remark: $\mathbb{R}^n$ is itself both open & closed.

eg: 1. $[a,b] \subseteq \mathbb{R}$ is a closed set.

$\mathbb{R} \setminus [a,b] = (-\infty, a) \cup (b, \infty)$

2. The set $\{(x,x) \in \mathbb{R}^2\}$ is closed in $\mathbb{R}^2$ wrt Euclidean metric.

3. The set $\{(x, 1/x, 1/x^2, x^3) \in \mathbb{R}^4 : x \neq 0\}$ is closed in $\mathbb{R}^4$.

Thm: Any two norms on $\mathbb{R}^n$ are equivalent.
That is, if $\|.\|$ & $\|.\|'$ are norms in $\mathbb{R}^n$, then
$\exists\ \alpha, \beta > 0$ s.t $\alpha\|x\| < \|x\|' < \beta\|x\|\ \forall\ x \in \mathbb{R}^n$

Ppn: Let $d$ & $d'$ be matrices on $\mathbb{R}^n$ given by norms.
Then a subset $U$ of $\mathbb{R}^n$ is open in $d$ iff it is open in $d'$

Pf: For a vec. $u \in \mathbb{R}^n$ & $r > 0$, let $B_u(r)$ denote the open ball in metric $d$ & let $B'_u(r)$ denote the open ball in metric $d'$.

Suppose $U$ is open in $\mathbb{R}^n$ for the metric $d$.
Then for each $u \in U$, $\exists\ r > 0$ s.t $B_u(r) \subseteq U$.

$$B_u(r) = \{x \in \mathbb{R}^n : d(x, u) < r\}$$

If $d$ is given by $\|.\|$, then

$$B_u(r) = \{x \in \mathbb{R}^n : \|x - u\| < r\}$$

If $d'$ is given by $\|.\|'$, then by the prev. thm,
$\exists\ \alpha, \beta > 0$ s.t $\alpha\|x-u\|' < \|x-u\| < \beta\|x-u\|'$

Then $B'_u(r/\beta) \subseteq U$ for the pt. $u \in U$.

$= \{ y \in \mathbb{R}^n : \|y-u\|' < r/\beta \}$

If $y \in B'_u(r/\beta)$, then $\|y-u\|' < r/\beta$

$\Rightarrow \beta \|y-u\|' < r$

$\Rightarrow \|y-u\| < r$

Thus whenever $B_u(r) \subseteq U$, $B_u(r/\beta) \subseteq U$ &

conversely whenever $B_u'(r) \subseteq U$, $B_u(\alpha r) \subseteq U$

This proves the proposition.

Inner product: An inner prod. on $\mathbb{R}^n$ is a symm, bilinear, positive definite form on $\mathbb{R}^n$.

It is a function

$\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ s.t

1. $\langle x, y \rangle = \langle y, x \rangle \; \forall \; x, y \in \mathbb{R}^n$
2. $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle \; \forall \; x, y, z \in \mathbb{R}^n$
3. $\langle x, x \rangle \geqslant 0 \; \forall \; x \in \mathbb{R}^n$ & $\langle x, x \rangle = 0$ iff $x = 0$.

Bounded set in $\mathbb{R}^n$: Assume that we have a metric $d$ on $\mathbb{R}^n$. Then a bounded subset of $\mathbb{R}^n$ is a subset $A$ which is contained in some $B_0(r)$.

eg: 1. $(-1,1)$ is bounded in $\mathbb{R}$.

$(-1,1) \subseteq B_0(2)$

2. $[5, 2029]$ is bounded, $[5, 2029] \subseteq B_0(2029.1)$

Compact set: A subset $V$ of $\mathbb{R}^n$ is called compact if it is closed & bounded.

Equivalently, a subset $V$ of $\mathbb{R}^n$ is compact if any open cover of $V$ has a finite subcover.

Open cover means that we have a coll. of open sets whose union contains the given set.

$\{U_\alpha\}_{\alpha \in \Lambda}$ of open sets s.t $V \subseteq \bigcup_{\alpha \in \Lambda} U_\alpha$

A finite subcover means a finite subset $\{\alpha_1, \dots, \alpha_n\}$ for some $n$, of $\Lambda$ s.t $V \subseteq U_{\alpha_1} \cup U_{\alpha_2} \dots \cup U_{\alpha_n}$

The equivalence of these two definitions of compact sets is an imp. thm.

eg: 1. $[a,b] \subseteq \mathbb{R}$ is compact

$\mathbb{R} \setminus [a,b] = (-\infty, a) \cup (b, \infty)$ is open

Hence $[a,b]$ is closed.

Also, $[a,b] \subseteq B_0(|a|+|b|+1)$

2. Any finite set in $\mathbb{R}^n$ is compact.

3. $\mathbb{Q} \subseteq \mathbb{R}$ is not compact as it is unbounded

4. $\mathbb{Q} \cap [a,b] = \{x \in \mathbb{Q} : a \leq x \leq b\}$ is not compact as it is not closed.

## Heine-Borel Thm

$[a,b] \subseteq \mathbb{R}$ is compact by the finite subcover def$^n$.

**Pp$^n$**: If $V \subseteq \mathbb{R}^n$ is compact by finite subcover def$^n$, then it is compact by closed & bounded ppt.

**Pf**: For any pt. $v \in V$, we have an open ball $B_0(\|v\|+1)$ that contains $v$.

Then $V \subseteq \bigcup_{v \in V} B_0(\|v\|+1)$

This is an open cover of $V$.

Then, by the def$^n$.

$$V \subseteq B_0(\|v_1\|+1) \cup \ldots \cup B_0(\|v_n\|+1), \quad \text{for some } v_1, \ldots, v_n \in V$$

If $\|v\| = \max_{1 \leq i \leq n} \|v_i\|$, then $V \subseteq B_0(\|v\|+1)$

T'fore $V$ is bounded.

Now, we prove that $\mathbb{R}^n \setminus V$ is open.

Let $u \in \mathbb{R}^n \setminus V$.
For any $v \in V$, we can find two open sets $U_v$ & $U_v'$
s.t $v \in U_v$ & $u \in U_v'$ & $U_v \cap U_v' = \emptyset$

This gives an open cover of $V \subseteq \bigcup_{v \in V} U_v$

Then, by def$^n$, $V \subseteq U_{v_1} \cup \dots \cup U_{v_n}$

Check that corresponding $U_{v_i}'$ each contain $u$.
T'fore $u \in \bigcap_{i=1}^{n} U_{v_i}'$.

This is an open set containing $u$ & contained in $\mathbb{R}^n \setminus V$.

Pp$^n$: Let $V \subseteq \mathbb{R}$ be closed and bounded. If $\{U_\alpha\}$ is an open cover of $V$, then there is a finite subcover of $V$.

Pf: Since $V$ is bounded, $V \subseteq B_0(r)$ for some $r$.

But $V \subseteq \mathbb{R}$, so $B_0(r) = (-r, r)$

$$V \subseteq (-r, r) \subseteq [-(r+1), r+1]$$

Observe, $V$ is closed, hence $\mathbb{R} \setminus V$ is open.

Call $\mathbb{R} \setminus V = U$

Then, $\{U_\alpha\}_{\alpha \in \Lambda} \cup \{U\}$ is an open cover of $[-(r+1), r+1]$

By def$^n$, there is a finite subcover.

$$[-(r+1), r+1] \subseteq U_{\alpha_1} \cup U_{\alpha_2} \cup \dots \cup U_{\alpha_n} \cup U$$

Then, $V \subseteq U_{\alpha_1} \cup \dots \cup U_{\alpha_n}$

Continuous fn$^n$: Let $U \subseteq \mathbb{R}^n$ be open,

A fn$^n$ $f : U \to \mathbb{R}^m$ is said to be cont. at $u \in U$ if for every $\epsilon > 0$, there is a $\delta > 0$, s.t

$$\|u - x\| < \delta \Rightarrow \|f(u) - f(x)\| < \epsilon \quad \forall\, x \in U$$

This means that if $A$ is an open subset of $\mathbb{R}^m$ containing $f(u)$, then

$$f^{-1}(A) = \{ x \in U : f(x) \in A \}$$

is an open subset of $\mathbb{R}^m$

A fn$^n$ $f : U \to \mathbb{R}^m$ is said to be cont. if it is cont. at every $u \in U$.

Rem: 1. The image of a closed set under a cont. fn need not be closed.

2. The image of a bnd. subset under a cont. fn need not be bnd.

3. The image of a compact subset under a cont. fn is compact.

eg: 1. $\pi_1 : \mathbb{R}^2 \to \mathbb{R}$,
$(x,y) \mapsto x$

The image of $\{(x,y) \in \mathbb{R}^2 : xy = 1\}$ is $(-\infty, 0) \cup (0, \infty)$

2. Consider $f : (0,1) \to \mathbb{R}$
$x \mapsto 1/x$

Given any subset $A \subseteq \mathbb{R}^n$, we want to approximate it by

1. open sets
2. closed sets.

However, arbitrary union of closed sets is <u>not</u> necessarily closed.

Take all open sets in $A$ & take their union. This is the largest open set in $A$. It is called the interior of $A$ & is denoted by $\operatorname{int} A$ or $A^\circ$.

Take all closed sets containing $A$ & take their intersection. This is the smallest closed set containing $A$. This is called the closure of $A$ & is denoted by $\bar{A}$.

$$A^\circ \subseteq A \subseteq \bar{A}$$

Exterior of $A$ = Interior of $\mathbb{R}^n \setminus A$

The boundary of A is the set of all points in $\bar{A}$ that are not in A. There are pts. admitting seq. both in A & $\mathbb{R}^n - A$ converging to them.

Let $f : A \to \mathbb{R}^m$ be a bound. fxn.
$(A \subseteq \mathbb{R}^n)$

Let $a \in A$ & $\delta > 0$.

Let $M(f, a, \delta) = \sup \{ f(x) : \|x - a\| < \delta, x \in A \}$
$m(f, a, \delta) = \inf \{ f(x) : \|x - a\| < \delta, x \in A \}$

$$O(f, a) = \lim_{\delta \to 0} \| M(f, a, \delta) - m(f, a, \delta) \|$$

This is called the oscillation of f at a.

Note that the above lim. exists as $M(f, a, \delta) - m(f, a, \delta)$ decreases as $\delta$ decreases.

Ppn: Let $f : A \rightarrow \mathbb{R}^m$ be bnd. & $a \in A$

$(A \subseteq \mathbb{R}^n)$

Then $f$ is cont. at $a$ iff $O(f, a) = 0$.

Pf: $(\Rightarrow)$ Assume that $f$ is cont. at $a$.

So, $\forall \epsilon > 0, \exists \delta > 0, \quad \|x - a\| < \delta \Rightarrow \|f(x) - f(a)\| < \epsilon$

Then, $\|M(f, a, \delta) - m(f, a, \delta)\| < 2\epsilon$

Indeed, for any two $x_1, x_2 \in B_a(\delta)$

$$\|f(x_1) - f(x_2)\| \leq \|f(x_1) - f(a)\| + \|f(x_2) - f(a)\|$$
$$< 2\epsilon$$

This inequality holds, probably with equality sign, for the limits.

Then as $\epsilon \rightarrow 0$, $\delta$ also goes to $0$ & hence $O(f, a) = 0$.

$$O(f, a) = \lim_{\delta \to 0} \|M(f, a, \delta) - m(f, a, \delta)\| = 0$$

($\Leftarrow$) Assume that $O(f,a) = 0$.

Let $\epsilon > 0$ be given.

Then $\exists\ \delta > 0$ s.t $\|M(f,a,\delta) - m(f,a,\delta)\| < \epsilon$

Since $\forall\ |x-a| < \delta$, $\quad m(f,a,\delta) \leq f(x) \leq M(f,a,\delta)$,

$$\Rightarrow \|f(x) - f(a)\| \leq \|M(f,a,\delta) - m(f,a,\delta)\| < \epsilon$$

As $\epsilon$ was arbitrary, $\lim_{x \to a} f(x) = f(a)$

## Derivative

Let $U \subseteq \mathbb{R}^n$ be an open set & let $f: U \to \mathbb{R}^m$ be a fxn. We say that $f$ is diff. at $a \in U$ if there is a linear map $\lambda: \mathbb{R}^n \to \mathbb{R}^m$ s.t

$$\lim_{x \to a} \frac{\|f(x) - f(a) - \lambda(x-a)\|}{\|x-a\|} = 0$$

Recall that a fxn $f: (a,b) \to \mathbb{R}$ is said to be diff. at $c \in (a,b)$ if $\lim_{x \to c} \frac{\|f(x) - f(c)\|}{\|x-c\|}$ exists

If this lim. exists, then it is a real no., denoted by $f'(c)$.
Then the existence of above lim. is equiv. to

$$\lim_{x \to c} \frac{\|f(x) - f(c) - f'(c)(x-c)\|}{\|x-c\|} \text{ being zero.}$$

The set of linear maps $T: \mathbb{R} \to \mathbb{R}$ are given by $T(1) \in \mathbb{R}$.

eg:

1. Let $f: U \to \mathbb{R}^m$ be the constant map.
$(U \subseteq \mathbb{R}^n)$

$f(u) = \alpha \in \mathbb{R}^m \ \forall \ u \in \mathbb{R}^n$.

To check if this $f$ is diff., we need to find a linear $\lambda: \mathbb{R}^n \to \mathbb{R}^m$ s.t $\displaystyle\lim_{x \to a} \frac{\|f(x) - f(a) - \lambda(x-a)\|}{\|x-a\|} = 0$

The LHS is, $\displaystyle\lim_{x \to a} \frac{\|\alpha - \alpha - \lambda(x-a)\|}{\|x-a\|} = \lim_{x \to a} \frac{\|\lambda(x-a)\|}{\|x-a\|}$

The zero map works for $\lambda$, $\forall \ a \in U$

T'fore $f$ is diff. at every pt. in $U$.

2. $m = n$, $f: U \to \mathbb{R}^m = \mathbb{R}^n$
$(U \subseteq \mathbb{R}^n)$
$u \mapsto u$

Then $\lambda = Id$ works

This fn is also diff.

Q. Is the fnⁿ $\lambda$ unique for a given pt. a?

Thm: If $\underset{(\subseteq \mathbb{R}^n)}{f : U} \to \mathbb{R}^m$ is diff. at $a \in U$, then there is a unique linear map $\lambda$ satisfying the defⁿ.

Pf: Assume that $\lambda_1$ & $\lambda_2$ are linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$ satisfying the defⁿ, for the given pt. $a \in U$. Then

$$\lim_{h \to 0} \frac{\|\lambda_1(h) - \lambda_2(h)\|}{\|h\|} = \lim_{h \to 0} \frac{\|(f(a+h) - f(a) - \lambda_2(h)) - (f(a+h) - f(a) - \lambda_1(h))\|}{\|h\|}$$

$$\leq \lim_{h \to 0} \frac{\|f(a+h) - f(a) - \lambda_2(h)\|}{\|h\|} + \lim_{h \to 0} \frac{\|f(a+h) - f(a) - \lambda_1(h)\|}{\|h\|}$$

$$= 0$$

Now, for any vec. $v \in \mathbb{R}^n$, $v \neq 0$, $\lim\limits_{t \to 0} tv = 0$

Then, $$\lim_{\substack{t \to 0 \\ t \in \mathbb{R}}} \frac{\|\lambda_1(tv) - \lambda_2(tv)\|}{\|tv\|} = \lim_{\substack{tv \to 0 \\ tv \in \mathbb{R}^n}} \frac{\|\lambda_1(tv) - \lambda_2(tv)\|}{\|tv\|} = 0$$

$$\text{LHS} = \lim_{t \to 0} \frac{\|t\lambda_1(v) - t\lambda_2(v)\|}{\|tv\|} = \lim_{t \to 0} \frac{\cancel{|t|}\,\|\lambda_1(v) - \lambda_2(v)\|}{\cancel{|t|}\,\|v\|} = 0$$

Since, $v \neq 0$, $\|v\| \neq 0 \Rightarrow \|\lambda_1(v) - \lambda_2(v)\| = 0$, then $\lambda_1(v) = \lambda_2(v)$

Using this thm, we conclude that $Df(a) = \lambda$ is a well-defined qty. This is called the derivative of $f$ at $a$.

Thm: Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be diff. at $a \in \mathbb{R}^n$ & let $g : \mathbb{R}^m \to \mathbb{R}^p$ be diff. at $b = f(a)$. Then, $g \circ f : \mathbb{R}^n \to \mathbb{R}^p$ is diff. at $a$

& $D(g \circ f)(a) = Dg(f(a)) \cdot Df(a)$

$$\mathbb{R}^n \xrightarrow{\underbrace{Df(a)}_{(m \times n)}} \mathbb{R}^m \xrightarrow{\underbrace{Dg(f(a))}_{(p \times m)}} \mathbb{R}^p$$

$$\underbrace{Dg(f(a))}_{(p \times m)} \cdot \underbrace{Df(a)}_{(m \times n)} : \mathbb{R}^n \to \mathbb{R}^p$$

Pf: Consider $\displaystyle \lim_{h \to 0} \frac{\| (g \circ f)(a+h) - (g \circ f)(a) - (Dg(f(a)) \cdot Df(a))(h) \|}{\|h\|}$

$$= \lim_{\substack{h \to 0 \\ h \in \mathbb{R}^n}} \frac{\| g(f(a+h)) - g(f(a)) - (Dg(f(a)) \cdot Df(a))(h) \|}{\|h\|} \quad -(*)$$

Since $Df(a)$ is a linear map, $\exists M > 0$ s.t $\|Df(a)(h)\| < M\|h\|$

We also have $\lim\limits_{h\to 0} \dfrac{\|f(a+h)-f(a)-Df(a)(h)\|}{\|h\|} = 0$

Let $\varphi(h) = f(a+h)-f(a)-Df(a)(h)$, then we get $\lim\limits_{h\to 0}\dfrac{\|\varphi(h)\|}{\|h\|}=0$ and $(*)$ becomes

$$\lim_{h\to 0}\frac{\|g(f(a)+\varphi(h)+Df(a)(h))-g(f(a))-Dg(f(a))\cdot(Df(a)(h))\|}{\|h\|}$$

$$= \lim_{h\to 0}\frac{\begin{array}{l}\|g(f(a)+\overbrace{[\varphi(h)+Df(a)(h)]}^{h'})-g(f(a))\\ -Dg(f(a))(Df(a)(h))+Dg(f(a))(\varphi(h))-Dg(f(a))(\varphi(h))\|\end{array}}{\|h\|}$$

$$= \lim_{h\to 0}\frac{\|g(f(a)+h')-g(f(a))-Dg(f(a))(h')+Dg(f(a))(\varphi(h))\|}{\|h\|}$$

$$\leq \lim_{h\to 0}\frac{\|g(f(a)+h')-g(f(a))-Dg(f(a))(h')\|}{\|h\|}$$

$$+ \lim_{h\to 0}\underbrace{\frac{\|Dg(f(a))(\varphi(h))\|}{\|h\|}}_{\leq \frac{|M|\,\|\varphi(h)\|}{\|h\|}\ \to\ 0 \text{ as } h\to 0}$$

$$= \lim_{h\to 0}\underbrace{\frac{\|g(f(a)+h')-g(f(a))-Dg(f(a))(h')\|}{\|h'\|}}_{\to 0 \text{ if } h'\to 0}\cdot\overbrace{\frac{\|h'\|}{\|h\|}}^{\text{exists as } h\to 0}$$

this holds if $h\to 0$

Consider the projection map $\pi_i : \mathbb{R}^n \to \mathbb{R}$

$$\pi_i(x_1, \dots, x_n) = x_i$$

This map is diff. at all pts. $a \in \mathbb{R}^n$. Further for any $a \in \mathbb{R}^n$ then $D\pi_i(a)$ is the $1 \times n$ matrix $[0, \dots, \underset{\substack{\uparrow \\ i\text{th place}}}{1}, \dots, 0]$

$$D\pi_i(a) : \mathbb{R}^n \to \mathbb{R}$$
$$(b_1, \dots, b_n) \mapsto b_i$$

Ppn: If $f : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map then $f$ is diff. at every $a \in \mathbb{R}^n$ & $Df(a) = f$

Pf: Consider $\displaystyle\lim_{h \to 0} \frac{\|f(a+h) - f(a) - f(h)\|}{\|h\|} = 0$

Note: As a varies in $\mathbb{R}^n$, the derivative $Df$ does not vary. Hence, the second derivative $D(Df)(a) = 0$

$Df$ is a map which takes input from $\mathbb{R}^n$ & returns an elem. in $\text{Hom}(\mathbb{R}^n, \mathbb{R}^m) = m \times n$ matrices.

Ppn: Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be given by $f(x) = (f_1(x), f_2(x), \ldots, f_m(x))$ & $a \in \mathbb{R}^n$. The $f$ is diff. at $a$ iff each $f_i$ is diff. at $a$.

Pf: For each $i$, $f_i = \pi_i \circ f$. Hence if $f$ is diff. at $a$, then by chain rule, each $f_i$ is diff. at $a$.

Further, $Df_i(a) = D(\pi_i \circ f)(a) = (D\pi_i)(f(a)) \cdot Df(a)$

$$= \pi_i \circ Df(a)$$

$=$ ith row of $m \times n$ matrix $Df(a)$.

Consistency check: $f_i : \mathbb{R}^n \to \mathbb{R}$, hence $Df_i(a)$ should be a $1 \times n$ matrix.

Conversely, we assume that each $f_i$ is diff. at $a \in \mathbb{R}^n$ & prove that $f$ is diff. at $a$.

Consider the linear map $\lambda : \mathbb{R}^n \to \mathbb{R}^m$, $\lambda(a) = (\lambda_1(a), \ldots, \lambda_m(a))$ for $\lambda_i(a) = Df_i(a)$.

$$\lambda(a) = [\lambda_1(a) \quad \ldots \quad \lambda_m(a)]^T \rightarrow m \times n \text{ matrix}$$

Now, we take the lim.,

$$\lim_{h\to 0}\frac{\|f(a+h)-f(a)-\lambda(h)\|}{\|h\|}=\lim_{h\to 0}\frac{\|(f_1(a+h),\dots,f_m(a+h))-(f_1(a),\dots,f_m(a))-(\lambda_1(a),\dots,\lambda_m(a))\|}{\|h\|}$$

$$=\lim_{h\to 0}\frac{\|(f_1(a+h)-f_1(a)-\lambda_1(a),\ \dots,\ f_m(a+h)-f_m(a)-\lambda_m(a))\|}{\|h\|}$$

$$\leq\sum_{i=1}^{m}\lim_{h\to 0}\frac{\|f_i(a+h)-f_i(a)-\lambda_i(a)\|}{\|h\|}=0$$

---

Consider the inclusions $\iota_j:\mathbb{R}\to\mathbb{R}^n$,

$$\alpha\mapsto(0,\dots,\underset{\substack{\uparrow\\ j\text{th place}}}{\alpha},\dots,0)$$

This map is linear, hence diff. at each $\alpha\in\mathbb{R}$, for each $j$.

The derivative is the $n\times 1$ matrix. $[0\ \dots\ 1\ \dots\ 0]^T$

Pp$^n$: Let $f:\mathbb{R}^n\to\mathbb{R}^m$ is diff. at $a\in\mathbb{R}^n$. Let $a=(a_1,\dots,a_n)$.

Consider the map $\iota_j:\mathbb{R}\to\mathbb{R}^n$, $\alpha\mapsto(a_1,\dots,a_{j-1},\alpha,a_{j+1},\dots,a_n)$.

Then for each $1\leq i\leq m$ & $1\leq j\leq n$, the map $\pi_i\circ f\circ\iota_j$ is diff. at $a_j$ &

$$D(\pi_i\circ f\circ\iota_j)(a_j)=\underbrace{D\pi_i(f(a))}_{D\pi_i(f(\iota_j(a_j)))}\cdot\underbrace{Df(a)}_{Df(\iota_j(a_j))}\cdot D\iota_j(a_j)$$

Note that, $\pi_i \circ f \circ g_j$ is from $\mathbb{R}$ to $\mathbb{R}$.

Indeed $\underbrace{D\pi_i(f(a))}_{1\times m} \cdot \underbrace{Df(a)}_{m\times n} \cdot \underbrace{Dg_j(a_j)}_{n\times 1}$ is a $1\times 1$ matrix.

<u>Cor</u>: If $f$ is diff. at $a = (a_1, \dots, a_n)$, then for every $i, j$, $\pi_i \circ f \circ g_j$ are diff. at each $a_j$.

These are nothing but the partial derivatives $\frac{\partial f_i}{\partial x_j}$

Check whether

$$f(x) = \begin{cases} 0, & x \in \mathbb{Q} \\ 1/q, & \text{if } x = p/q \text{ in reduced form} \end{cases}$$

&

$$f(x) = \begin{cases} \dfrac{x^3 + xy^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

has partial derivatives & whether it is diff. at $(0,0)$.

Thm: Let $f: \mathbb{R}^n \to \mathbb{R}^m$ be a fnⁿ & let $a = (a_1, \dots, a_n) \in \mathbb{R}^n$

Then if all $\partial f_i / \partial x_j$ exist at $a_j$ & all are cont. at $a_j$, then $f$ is diff. at $a$.

Such $\pi_i \circ f \circ g_j$ are called continuously differentiable.

Pf: It is sufficient to prove that each $\pi_i \circ f$ is diff. at $a$. We therefore assume that $m = 1$.

We have that $\partial f / \partial x_j$ exist & is cont. diff. at $a_j$ for each $j$.

Def. a matrix, $\lambda = [\partial f / \partial x_j]_{1 \times n}$

$$= \left[ \frac{\partial f}{\partial x_1}(a_1), \dots, \frac{\partial f}{\partial x_n}(a_n) \right]$$

Consider, $\lim\limits_{h \to 0} \dfrac{\| f(a+h) - f(a) - \lambda(h) \|}{\| h \|}$ & let $h = (h_1, \dots, h_n)$

$$\Rightarrow \lim_{h \to 0} \frac{\| f(a_1 + h_1, \dots, a_n + h_n) - f(a_1, \dots, a_n) - \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(a_j) h_j \|}{\| h \|}$$

Note that, $f(a+h) - f(a) = f(a_1+h_1, \dots, a_n+h_n)$

$$- f(a_1+h_1, \dots, a_{n-1}+h_{n-1}, a_n)$$
$$+ f(a_1+h_1, \dots, a_{n-1}+h_{n-1}, a_n)$$
$$\vdots$$
$$+ f(a_1+h, a_2, \dots, a_n) - f(a_1, a_2, \dots, a_n)$$

By MVT in one dimension, we get that for each $j$ $\exists\, \alpha_j \in (a_j, a_j+h_j)$ s.t

$$f(a_1+h_1, \dots, a_j+h_j, a_{j+1}, \dots, a_n) - f(a_1+h_1, \dots, a_j, \dots, a_n) = \frac{\partial f}{\partial x_j}(\alpha_j) \cdot h_j$$

Then, $$\lim_{h \to 0} \frac{\| f(a+h) - f(a) - \lambda(h) \|}{\|h\|}$$

$$= \lim_{h \to 0} \frac{\| \sum_j \frac{\partial f}{\partial x_j}(\alpha_j) \cdot h_j - \sum_j \frac{\partial f}{\partial x_j}(a_j) \cdot h_j \|}{\|h\|}$$

$$\leq \lim_{h \to 0} \frac{\sum_j \left| \frac{\partial f}{\partial x_j}(\alpha_j) - \frac{\partial f}{\partial x_j}(a_j) \right| \cdot |h_j|}{\|h\|}$$

$$\leq \lim_{h \to 0} \sum_j \left| \frac{\partial f}{\partial x_j}(\alpha_j) - \frac{\partial f}{\partial x_j}(a_j) \right| \qquad (\because |h_j| / \|h\| < 1)$$

As $h \to 0$, each $h_j \to 0$ & hence $\alpha_j \to a_j$

Since each $\partial f / \partial x_j$ is cont. at $a_j$, we get $\lim_{h \to 0} \frac{\| f(a+h) - f(a) - \lambda(h) \|}{\|h\|} = 0$

Hence, $f$ is diff. at $a$ & $Df(a) = \lambda$

**Recall**: If $f:(a,b)\to\mathbb{R}$ is cont. diff. on $(a,b)$. If $\exists\, \alpha\in(a,b)$ s.t $f'(\alpha)\neq 0$. Then $\exists$ a nbd $(a_1,b_1)\subseteq(a,b)$ of $\alpha$ s.t $f'$ does **not** vanish on $(a_1,b_1)$.

Then $f$ is one-one on $(a_1,b_1)$ & hence invertible.

We then have that $f^{-1}$ is diff. at $\alpha$ & $D(f^{-1}(\alpha)) = \dfrac{1}{Df(a)}$

**Lem**: Let $A\subseteq\mathbb{R}^n$ be a rectangle. Let $f:A\to\mathbb{R}^n$ be cont. diff. If there is $M>0$ s.t $\left|\dfrac{\partial f_i}{\partial x_j}\right| < M$ on $A$, then

$$\|f(x)-f(y)\| < n^2 M\|x-y\| \text{ for } x,y\in A.$$

**Pf**: Consider

$$\begin{aligned} f_i(x)-f_i(y) &= f_i(x_1,\dots,x_n) - f_i(y_1,\dots,y_n) \\ &= f_i(x_1,\dots,x_{n-1},x_n) - f_i(x_1,\dots,x_{n-1},y_n) \\ &+ f_i(x_1,\dots,x_{n-1},y_n) - f_i(x_1,\dots,y_{n-1},y_n) \\ &\;\;\vdots \\ &+ f_i(x_1,y_2,\dots,y_n) - f_i(y_1,y_2,\dots,y_n) \end{aligned}$$

$$\Rightarrow f_i(x)-f_i(y) = (x_n-y_n)\frac{\partial f_i}{\partial x_n}(z_n) + \dots + (x_1-y_1)\frac{\partial f_i}{\partial x_1}(z_1), \quad z_k\in(y_k,x_k)\ \forall\, 1\le k\le n$$

Then $|f_i(x) - f_i(y)| \leq \sum_{j=1}^{n} |x_j - y_j| \cdot \left| \frac{\partial f_i}{\partial x_j}(z_j) \right| \leq \sum_{j=1}^{n} |x_j - y_j| \cdot M$

$$\leq M \sum_{j} \|x-y\| = nM\|x-y\|$$

$$\|f(x) - f(y)\| \leq \sum_i |f_i(x) - f_i(y)| \leq n^2 M \|x-y\|$$

Thm: (Inverse fxn thm)

Let $f: \mathbb{R}^n \to \mathbb{R}^n$ be cont. diff. at $a \in \mathbb{R}^n$. Assume that $\det Df(a) \neq 0$. Then there are open sets $V$ containing $a$, $W$ containing $f(a)$ s.t $f: V \to W$ is one-one & onto and $\exists$ cont. fxn $g: W \to V \subseteq \mathbb{R}^n$ with $f \circ g = Id_W$, $g \circ f = Id_V$. Further $g$ is diff. at $f(a)$ with $Dg(f(a)) = (Df(a))^{-1}$.

Pf:

Step 1: Let $\lambda = Df(a)$. We have that $\lambda$ is invertible. Consider the fxn $h = \lambda^{-1} \circ f$.

$$Dh(a) = \underbrace{D\lambda^{-1}(f(a))}_{\lambda^{-1}} \cdot \underbrace{Df(a)}_{\lambda} = Id$$

Moreover, if we prove the result for $\lambda^{-1} \circ f$, then the result for $f$ will follow.

We therefore work with the special case that $Df(a) = Id$.

We have $\lim_{h \to 0} \frac{\| f(a+h) - f(a) - h \|}{\|h\|} = 0$

If we choose $\epsilon = 1$, $\exists \delta > 0$ s.t for $\|h\| < \delta$,

$$\frac{\| f(a+h) - f(a) - h \|}{\|h\|} < 1$$

T'fore for $\|h\| < \delta$, $f(a+h)$ is never equal to $f(a)$.
(unless $h = 0$)

We then have a rect. $A_1$ containing $a$ s.t inside $A_1$
$f(x) = f(a)$ only if $x = a$.

Further, $Df(a)$ is invertible, i.e $\det Df(a) \neq 0$ & since $Df$ is cts at $a$, we get that there is a rect. $A_2$ containing $a$ s.t $Df$ is inv. for every $x \in A_2$.

Finally, using continuity of the partial derivatives, we get a rect. $A_3$ containing $a$ s.t $\left\| \frac{\partial f_i(x)}{\partial x_j} - \frac{\partial f_i(a)}{\partial x_j} \right\| < \frac{1}{2n^2}$

for $x \in A_3$

Wlog, $A_1 = A_2 = A_3 = A$

We have a rect. A containing a s.t.

1. $f(x) \neq f(a)$ for $x \in A$, $x \neq a$
2. $Df(a)$ is invertible for $x \in A$
3. $\left\| \frac{\partial f_i}{\partial x_j}(x) - \frac{\partial f_i}{\partial x_j}(a) \right\| < \frac{1}{2n^2}$ for $x \in A$

Consider the fxn $g(x) = f(x) - x$.

$$\left\| \frac{\partial g_i(x)}{\partial x_j} \right\| = \left\| \frac{\partial f_i}{\partial x_j}(x) - \delta_{ij} \right\| < \frac{1}{2n^2} \quad \forall\ x \in A$$

Then by the prev. lemma,

$$\| f(x) - x - f(y) + y \| < \frac{1}{2} \| x - y \| \quad \text{for } x, y \in A$$

Further, $\|x-y\| - \|f(x) - f(y)\| < \|(f(x) - f(y)) - (x - y)\|$

$$< \frac{1}{2} \|x - y\|$$

$\Rightarrow \|x-y\| < 2\|f(x) - f(y)\| \quad \forall\ x, y \in A.$

This proves then f is one-one on A.

Now consider the b'ry of A. This is compact, $a \notin \partial A$, then $f(a) \notin f(\partial A)$

Then, the dist. of $f(a)$ from $f(\partial A)$, $\delta$, is positive.

Def. $W = B_{f(a)}(\delta/2)$.

Def. $V = \underbrace{f^{-1}(W)}_{\text{open}} \cap \underbrace{\mathring{A}}_{\text{open}}$, so $V$ is an open subset inside $A$ containing $a$. The fn $f: V \to W$ is one-one & onto.

Thus $f^{-1}: W \to V$ exists.

The inequality above says that $\|f^{-1}(w_1) - f^{-1}(w_2)\| < 2\|w_1 - w_2\|$.

Thus $w_1 \to w_2$ forces $f^{-1}(w_1) \to f^{-1}(w_2)$.

Hence $f^{-1}$ is cts.

Fix $w \in W$ & let $v = f^{-1}(w) \in V$. Since $f$ is diff. at $v$, we get

$$\lim_{h \to 0} \frac{\| f(v+h) - f(v) - \mu \cdot h \|}{\|h\|} = 0 \quad \text{for } \mu = Df(v).$$

We want to prove that $D^{-1}f(w) = \mu^{-1}$

Consider $\lim\limits_{k \to 0} \dfrac{\|f^{-1}(w+k) - f^{-1}(w) - \mu^{-1} k\|}{\|k\|}$

Consider $\chi(k) = f^{-1}(w+k) - f^{-1}(w) - \mu^{-1}(k)$

& $\varphi(h) = f(v+h) - f(v) - \mu(h)$

We have $\lim\limits_{h \to 0} \dfrac{\|\varphi(h)\|}{\|h\|} = 0$.

We want to prove that $\lim\limits_{k \to 0} \dfrac{\|\chi(k)\|}{\|k\|} = 0$

$$\lim_{k \to 0} \frac{\|\chi(k)\|}{\|k\|} = \lim_{k \to 0} \frac{\|f^{-1}(w+k) - f^{-1}(w) - \mu^{-1}(k)\|}{\|f^{-1}(w+k) - f^{-1}(w)\|} \cdot \frac{\|f^{-1}(w+k) - f^{-1}(w)\|}{\|k\|}$$

$$= \left( \lim_{h \to 0} \frac{\|v+h-v-\mu^{-1}(k)\|}{\|v+h-v\|} \right) \left( \lim_{k \to 0} \frac{\|f^{-1}(w+k) - f^{-1}(w)\|}{\|k\|} \right)$$

$$= \left( \lim_{h \to 0} \frac{\|h-\mu^{-1}(k)\|}{\|h\|} \right) \left( \lim_{k \to 0} \frac{\|f^{-1}(w+k) - f^{-1}(w)\|}{\|k\|} \right)$$

$k = (w+k) - w$ & $k = \mu(f^{-1}(w+k) - f^{-1}(w) - \chi(k))$

$h = (v+h) - v$ & $h = \mu^{-1}(f(v+h) - f(v) - \varphi(h))$

→ Fix $a \in W$. $\|w - f(a)\| < \|w - f(y)\| \quad \forall\, y \in \partial A$

Consider the fx$^n$

$$h(x) = \|w - f(x)\|^2 \quad \forall\, x \in A$$

$$= \sum_{i=1}^{n} (w_i - f_i(x))^2$$

This is diff.

Since A is compact, it attains a minimum value on A.

That is, $\exists\, x_0 \in A$ s.t $h(x_0)$ is the min. value on A.

Then at $x_0$, $h'(x_0) = 0$ which means that for every j.

$$\sum_{i=1}^{n} 2(w_i - f_i(x_0) \frac{\partial f_i(x_0)}{\partial x_j} = 0$$

This gives a matrix multip$^n$.

$$[2(w_i - f_i(x_0)]_{i\,\,1\times n} \left[\frac{\partial f_i(x_0)}{\partial x_j}\right]_{n\times n} = [0 \ \ldots \ 0]_{1\times n}$$

Since $\left[\frac{\partial f_i(x_0)}{\partial x_j}\right]$ is invertible, we get $w_i - f_i(x_0) = 0 \quad \forall i$

$\Rightarrow \quad w = f(x_0)$

Infact, $x_0 \in V$

Hence $f$ is onto.

Now, $\quad \|x-y\| \leq 2\|f(x) - f(y)\| \quad \forall x, y \in V$

This can be rewritten as $\|f^{-1}(w_1) - f^{-1}(w_2)\| \leq 2\|w_1 - w_2\| \quad \forall w_1, w_2 \in W$

Thus, $f^{-1}$ is cts.

Let $w \in W$ be fixed & let $v = f(w) \in V$

Since $f$ is diff. at $v$, we get for $\mu = Df(x)$,

$f(v+h) - f(v) - \mu h = \varphi(h)$ has the property that

$\frac{\|\varphi(h)\|}{\|h\|} \to 0$ as $h \to 0$

This is equivalent to $w+k-w = \mu(f^{-1}(w+h) - f^{-1}(w)) - \varphi(f^{-1}(w+h) - f^{-1}(w))$

Further, $\mu^{-1}((w+k)-w) - (f^{-1}(w+k)-f^{-1}(w)) = \mu^{-1}\varphi(f^{-1}(w+k)-f^{-1}(w))$

We def. $\chi$ by $\chi(k) = f^{-1}(w+k) - f^{-1}(w) - \mu^{-1}k = \mu^{-1}\varphi(f^{-1}(w+k)-f^{-1}(w))$

We want to show $\dfrac{\|\chi(k)\|}{\|k\|} \to 0$ as $k \to 0$.

Consider $\displaystyle\lim_{k\to 0} \frac{\|\mu^{-1}\varphi(f^{-1}(w+k)-f^{-1}(w))\|}{\|k\|}$

The limit is zero if $\displaystyle\lim_{k\to 0} \frac{\|\varphi(f^{-1}(w+k)-f^{-1}(w))\|}{\|k\|} = 0$

Here, $\displaystyle\lim_{k\to 0} \frac{\|\varphi(f^{-1}(w+k)-f^{-1}(w))\|}{\|w+k-w\|}$

$$= \lim_{k\to 0} \underbrace{\frac{\|\varphi(f^{-1}(w+k)-f^{-1}(w))\|}{\|f^{-1}(w+k)-f^{-1}(w)\|}}_{\substack{\downarrow \\ 0}} \cdot \underbrace{\frac{\|f^{-1}(w+k)-f^{-1}(w)\|}{\|(w+k)-w\|}}_{\text{bounded}}$$

Hence, $\displaystyle\lim_{k\to 0} \frac{\|\varphi(f^{-1}(w+k)-f^{-1}(w))\|}{\|k\|} = 0$

$\Rightarrow \quad \mu^{-1} = Df^{-1}(w)$

Implicit fn^n thm : Let $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ be ctsly diff. at $(a,b) \in \mathbb{R}^n \times \mathbb{R}^m$ with $f(a,b) = 0$. If the determinant $\det\left(\frac{\partial f_i}{\partial x_{n+j}}(a,b)\right) \neq 0$, then $\exists$ an open subset $A$ of $\mathbb{R}^n$ containing $a$, an open subset $B$ of $\mathbb{R}^m$ containing $b$ & a unique fn^n $g : A \to B$ s.t $f(x, g(x)) = 0 \ \forall \ x \in A$. Further $g$ is diff.

Pf : Def. $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^m$ by $F(x,y) = (\underbrace{x}_{(h_1, \dots, h_n)}, \underbrace{f(x,y)}_{(f_1, \dots, f_m)})$

Then $\det(DF(a,b)) = \det\left(\frac{\partial f_i}{\partial x_{n+j}}(a,b)\right)$

$$DF(a,b) = \left[\begin{array}{c|c} \frac{\partial h_i}{\partial x_j}{}_{n \times n} & \frac{\partial h_i}{\partial x_{n+j}} \\ \hline \frac{\partial f_i}{\partial x_j} & \frac{\partial f_i}{\partial x_{n+j}} \end{array}\right] = \left[\begin{array}{c|c} 1_{n \times n} & 0 \\ \hline * & \frac{\partial f_i}{\partial x_{n+j}} \end{array}\right](a,b)$$

$$\Rightarrow \det(DF(a,b)) = \det\left(\frac{\partial f_i}{\partial x_{n+j}}(a,b)\right) \neq 0$$

We can apply the inverse fn^n thm.

There is an open $V$ of $\mathbb{R}^n \times \mathbb{R}^m$ containing $(a,b)$ & an open $W$ inside $\mathbb{R}^n \times \mathbb{R}^m$ containing $(a,0)$ & a diff. $G : W \to V$ s.t $F \circ G = Id_W$, $G \circ F = Id_V$.

Take a basic open set in $V$ containing $(a,b)$ namely, $A\times B$, where $A$ is subset of $\mathbb{R}^n$ containing $a$ & $B$ is an open subset of $\mathbb{R}^m$ containing $b$.

$$F : (A\times B) \to \underbrace{F(A\times B)}_{W_1 = G^{-1}(A\times B)} \subseteq W$$

$$\begin{tikzcd} F : A\times B \arrow[r] & W_1 \arrow[l, bend left, "G|_{W_1}"] \end{tikzcd}$$

Note that, $G(u,v) = (u, g(u,v))$

$F(x,y) = (x, f(x,y))$ , $F(a,b) = (a,0)$

$G(u,v) = (u, g(u,v))$ $(a,0) \in W_1$

$F \circ G(u,v) = F(u, g(u,v)) = (u, f(u, g(u,v))) = (u,v)$

$\Rightarrow$ We have written $v$ as a $fn^n$ of $f$ & $g$ on a nbd of $0$.

If $v=0$, then we get $f(u, g(u,0)) = 0$

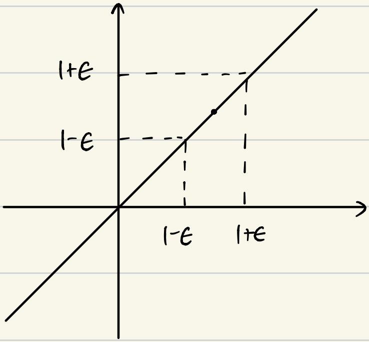
Hence, in a nbd of $a$, we found $a$.

$g : A \to B$ s.t $f(x, g(x)) = 0$

$a \mapsto b$

eg: $f: \mathbb{R}\times\mathbb{R} \to \mathbb{R}$

$(x,y) \mapsto y-x$



$(a,b)=(1,1)$

$F: \mathbb{R}\times\mathbb{R} \to \mathbb{R}\times\mathbb{R}$

$(x,y) \mapsto (\underset{u}{x}, \underset{v}{y-x})$

$$\det \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}_{(1,1)} = 1 \neq 0$$

$v = y-x \quad \Rightarrow \quad x = u$

$u = x \qquad\qquad y = v+u$

$$(u, v+u) \xleftarrow{G} (u,v)$$

Putting $v=0$, we get $(u,u)$.

So, we have solved for $y$ in terms of $x$.

**Thm**: Let $p \leq n$ & let $f: \mathbb{R}^n \to \mathbb{R}^p$ be ctsly diff. at $a \in \mathbb{R}^n$ with $f(a) = 0$. If $Df(a)$ has rank $p$, then there is an open set $A$ in $\mathbb{R}^n$ containing $a$ & a diff. $h: A \to \mathbb{R}^n$ s.t $f \circ h(x_1, \dots, x_n) = (x_{n-p+1}, \dots, x_n)$

**Pf**: Consider $F: \mathbb{R}^{n-p} \times \mathbb{R}^p \to \mathbb{R}^{n-p} \times \mathbb{R}^p$

$$F(x_1, \dots, x_n) = (x_1, \dots, x_{n-p}, \underbrace{f(x_1, \dots, x_n)}_{y_1, \dots, y_n})$$

$$DF(a_1, \dots, a_n) = \left[\begin{array}{c|c} I_{(n-p)\times(n-p)} & 0 \\ \hline \multicolumn{2}{c}{Df(a_1, \dots, a_n)_{p \times n}} \end{array}\right]$$

Assume that the last $p$ columns of $Df(a)$ are linearly indep. Then $DF(a)$ is also inv'ble.

Using inverse fxn thm, we get $G: \underset{\mathbb{R}^n}{\overset{\subseteq}{W}} \to \underset{\mathbb{R}^n}{\overset{\subseteq}{V}}$ diff. for $W$ an open subset of $\mathbb{R}^n$ containing $F(a) = b$, $V$ an open subset of $\mathbb{R}^n$ containing $a$ s.t $G(y_1, \dots, y_n) = (y_1, \dots, y_{n-p}, k(y_1, \dots, y_n))$

Then $f \circ G(y_1, \dots, y_n) = f(y_1, \dots, y_{n-p}, k(y_1, \dots, y_n))$

$$= (y_{n-p+1}, \dots, y_n)$$

Now, the last $p$ cols. of $Df(a)$ may not be linearly indep.
Let $1 \leq j_1 < j_2 < \dots < j_p \leq n$ be s.t. the $j_1, \dots, j_p$ cols. are linearly indep.

Then apply a permutation, $\sigma : \mathbb{R}^n \to \mathbb{R}^n$ that takes the $j_\ell$-th col. to $(n+p+\ell)$-th col.

This reduces to the case done above.

# Integration

Let $A = [a_1, b_1] \times \ldots \times [a_n, b_n]$ be a closed rect. in $\mathbb{R}^n$ & consider a fn $f : A \to \mathbb{R}$ which is bounded.

A partition of A will be a set of sub-rectangles whose union is A.

If P is a partition of A, then for each subrectangle S in P

$$m_f(S) = \inf\{ f(x) : x \in S \}$$
$$M_f(S) = \sup\{ f(x) : x \in S \}$$

The volume of S is the product of lengths of all its sides.

$$v([c_1, d_1] \times \ldots \times [c_n, d_n]) = \prod_{i=1}^{n} (c_i - d_i)$$

We define, $L(f, P) = \sum_{S \in P} m_f(S) \cdot v(S)$, the lower sum of f at P

$U(f, P) = \sum_{S \in P} M_f(S) \cdot v(S)$, the upper sum of f at P

Clearly, $L(f, P) \leq U(f, P)$

A refinement $P'$ of a partition $P$ of $A$ if every $S' \in P'$ is contained in some $S \in P$.

**Lem:** If $P'$ is a refinement of $P$, then

$$L(f,P) \leq L(f,P') \quad \& \quad U(f,P) \geq U(f,P')$$

**Pf:** For each $S' \in P'$, $m_f(S') \geq m_f(S)$ where $S' \subseteq S \in P$

Then $L(f,P') \geq L(f,P)$

Sim, we can show $U(f,P') \leq U(f,P)$

**Lem:** If $P, Q$ are partitions of $A$.

Then $L(f,P) \leq U(f,Q)$

**Pf:** Take the partition $P \cap Q$. This refines the partitions $P$ & $Q$.

Then, $L(f,P) \leq L(f, P\cap Q) \leq U(f, P \cap Q) \leq U(f,Q)$

**Cor:** $\sup L(f,P) \leq \inf U(f,P)$

Let $A$ be a rectangle in $\mathbb{R}^n$ & $f : A \to \mathbb{R}$ be a fn,
assume that $f$ is bounded.

Take a partition $P$ of $A$ by subrectangles. For each $S \in P$,
obtain $m_S(f) = \inf\{f(x) : x \in S\}$, $M_S(f) = \sup\{f(x) : x \in S\}$

We form $L(f, P) = \sum_{S \in P} m_S(f) v(S)$, $\quad U(f, P) = \sum_{S \in P} M_S(f) v(S)$.
We say that $f$ is <u>integrable</u> if

$$\sup_P L(f, P) = \inf_Q U(f, Q)$$

& this common value is defined to be the integral of $f$ over $A$. It is written as $\int_A f$.

<u>eg</u>: 1. Let $A = [0,1]^n \subseteq \mathbb{R}^n$.
Consider $f : A \to \mathbb{R}$, $f(x) = 2026 \ \forall x \in A$.
It is clear that for any partition $P$ of $A$ & for any subrectangle $S \in P$, $m_S(f) = 2026$ & $M_S(f) = 2026$.
Then $L(f, P) = 2026$ & $U(f, P) = 2026$.
Hence $\sup L(f, P) = 2026 = \inf U(f, P)$.
Thus $f$ is int'ble & $\int_A f = 2026$

2. Let $A = [0,1]^n \subseteq \mathbb{R}^n$ & def. $f(x) = \begin{cases} 0, & \text{if } x \in \mathbb{Q} \\ 1, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$

Take a partition $P$ of $A$ & choose $S \in P$. Then $\exists\, x \in S$ with $f(x) = 0$ & $y \in S$ with $f(y) = 1$. Then $m_S(f) = 0$ & $M_S(f) = 1$.

Then $L(f,P) = \sum_{S \in P} m_S(f) \cdot v(S) = \sum_{S \in P} 0 \cdot v(S) = 0$

$U(f,P) = \sum_{S \in P} M_S(f) \cdot v(S) = \sum_{S \in P} 1 \cdot v(S) = 1$

Thus $\sup L(f,P) = 0$ & $\inf U(f,P) = 1$

Hence, the fxn $f$ is <u>not</u> int'ble

<u>Ex</u>: Consider $A = [0,1] \times [0,1]$.

Def

$$f(x,y) = \begin{cases} 0, & x \notin \mathbb{Q} \\ 0, & x \in \mathbb{Q}, y \notin \mathbb{Q} \\ 1/b, & x \in \mathbb{Q}, y \in \mathbb{Q}, \ y = a/b \text{ in its lowest form} \end{cases}$$

<u>A</u>. For any partition $P$ of $A$ & any $S \in P$, $m_S(f) = 0$ & hence $L(f,P) = 0$.

To show that $f$ is int'ble, we need to show that $\inf U(f, P) = 0$

Note that for any partition $P$, $U(P, f) > 0$.
We will show that for any $\epsilon > 0$, $\exists$ a partition $P_\epsilon$ of $A$ s.t $U(f, P_\epsilon) < \epsilon$

There are finitely many subrectangles in any partition s.t $M_S(f) \geqslant \frac{1}{n}$ for any chosen $n$.

Let $\epsilon > 0$ be fixed.
Choose $N$ s.t $\frac{1}{N} < \epsilon/2$. For subrectangles $S$ in any partition $P$ with $M_S(f) \leq 1/N$, the sum of $M_S(f) \cdot V_S(S)$ will be $< \epsilon/2$

The other possible values of $M_S(f)$ are $\{1, 1/2, 1/3, \dots, 1/N-1\}$, so we need to worry abt the horizontal lines $y = j/i$ with $1/i \geqslant 1/N-1$. Assume that the no. of these lines is $m$.
Choose subintervals on $x = 0$, the $y$-axis, containing the pts. $(0, j/i)$ with $i \leq N-1$, s.t the sum of their lengths is $< \epsilon/2$.

This gives a partition of A in the following way:
Take the subrectangle formed by the above sub-intervals with the full horizontal line. Take the closures of connected components of the remaining part as the other subrectangles.

Call this as subset $P_2$. Then $\underbrace{P_1}_{M_S(f) \geq \frac{1}{N-1}} \cup \underbrace{P_2}_{M_S(f) < \epsilon/2}$ is the partition

$$U(f,P) = \sum_{S \in P_1} M_S(f) \cdot v(S) + \sum_{S \in P_2} M_S(f)\, v(S)$$

$$\sum_{S \in P_2} M_S(f)\, v(S) < \sum_{S \in P_2} \epsilon/2 \cdot v(S) = \epsilon/2 \sum_{S \in P_2} v(S) < \epsilon/2 \cdot 1 = \epsilon/2$$

$$\sum_{S \in P_1} M_S(f) \cdot v(S) \leq \sum_{S \in P_1} 1 \cdot v(S) = \sum_{S \in P_1} (\text{length of the y-part of } S) \cdot \epsilon/2$$

$$< \epsilon/2$$

Thus, $U(f, P_\epsilon) = \sum_{P_1} + \sum_{P_2} < \epsilon/2 + \epsilon/2 = \epsilon$

If we fix a $y \in \mathbb{Q}$ & consider $g : [0,1] \to \mathbb{R}$, $g(x) = f(x,y)$, then $g$ is <u>not</u> int'ble

We have $g(x) = \begin{cases} 0, & x \notin \mathbb{Q} \\ 1/q, & x \in \mathbb{Q} \end{cases}$

This fx^n is <u>not</u> int'ble since $L(g,P) = 0$ & $U(g,P) = 1/q$

Consider $h : [0,1] \times [0,1] \to \mathbb{R}$, $h(x,y) = \begin{cases} 0, & \text{if } x \notin \mathbb{Q} \text{ or } y \notin \mathbb{Q} \\ 1/q \cdot 1/b, & \text{if } x = p/q,\ y = a/b \\ & \text{in lowest form} \end{cases}$

Is this fx^n int'ble?

Note that for each horizontal or vertical slice, $h$ restricted to the slice is int'ble.

The fx^n $h$ is int'ble & $\int_{[0,1]^2} h = 0$

<u>Rem</u>: Def. $\chi_A : [0,1] \to \mathbb{R}$ where $A \subseteq [0,1]$ is dense subset s.t $[0,1] \setminus A$ is also dense.

Then $\chi_A(x) = \begin{cases} 0, & x \notin A \\ 1, & x \in A \end{cases}$ is <u>not</u> int'ble

$L(\chi_A, P) = 0$ & $U(\chi_A, P) = 1$

Take a partition $P$ of $[0,1]^2$. For any $S \in P$, $S \cap ((\mathbb{R}\setminus\mathbb{Q}) \times (\mathbb{R}\setminus\mathbb{Q})) \neq \emptyset$

Then $m_S(h) = 0$. Hence $L(h,P) = 0$ & $\sup_P L(h,P) = 0$

To show $h$ is int'ble, we want to show that $\inf_P U(h,P) = 0$

For any $\epsilon > 0$, we construct at partition $P_\epsilon$ of $[0,1]^2$ s.t $U(h, P_\epsilon) < \epsilon$.

Consider $A_1 = \{ (p/q, a/b) \in \mathbb{Q} \times \mathbb{Q} : 1/q \cdot 1/b < \epsilon/2 \}$

& $A_2 = \{ (p/q, a/b) \in \mathbb{Q} \times \mathbb{Q} : 1/q \cdot 1/b \geq \epsilon/2 \}$

$= \{ (p/q, a/b) \in \mathbb{Q} \times \mathbb{Q} : qb \leq 2/\epsilon \}$

Then $A_2$ is a finite set for each pt. in $A_2$, construct a rect. around it of size $\epsilon/2 \cdot 1/m$ where $m = |A_2|$

There are finitely many subrect. in $[0,1] \times [0,1]$.

We can adjust these rect. further, s.t the intersection of any two is not again a subrect. This can be done because we have a finite set.

For the remaining part of $[0,1] \times [0,1]$, just take any rect. that are needed to fill the part.

Take the first non-zero pt. in the $x$-axis which is the projection of some subrect. covering a pt. of $A_2$. After that, take the second pt. & so on ...

These give you vertical slices. Then work as in the prev. problem.

This gives us a partition $P_\epsilon$ where $U(h, P_\epsilon) = \sum_{S \in P_\epsilon} M_S(h) \cdot v(S)$

If $S_2$ is the set of subrect. covering $A_2$ & $S_1$ is the set of remaining subrect. then,

$$\sum_{S \in S_1} M_S(h) \cdot v(S) \leqslant \frac{\epsilon}{2} \sum_{S \in S_1} v(S) < \epsilon/2$$

$$\sum_{S \in S_2} M_S(h) \cdot v(S) \leqslant 1 \cdot \sum_{S \in S_2} v(S) < \epsilon/2$$

$$U(h, P_\epsilon) = \sum_{S \in S_1} M_S(h) \cdot v(S) + \sum_{S \in S_2} M_S(h) \cdot v(S)$$

$$< \epsilon/2 + \epsilon/2 = \epsilon$$

Let $A \subseteq [0,1] \times [0,1]$ be a countably infinite & def. $f$ on $[0,1] \times [0,1]$ s.t. $f(x) = 0$ if $x \notin A$ & $\lim f(x) = 0$ for $x \in A$. Prove that $f$ is int'ble & $\int_{[0,1]^2} f = 0$.

Def$^n$: Let $A \subseteq \mathbb{R}^n$. We say that A has measure 0 if for $\epsilon > 0$, there are closed rect. $V_i \subseteq \mathbb{R}^n$ covering A s.t $\sum v(V_i) < \epsilon$

Examples:

1. Any finite set has measure 0. If the set is $\{a_1, \dots, a_r\} \subseteq \mathbb{R}^n$, then cover each $\{a_i\}$, $a_i = (a_{i1}, a_{i2}, \dots, a_{in})$ by $\Pi_j = [a_{ij} - \delta, a_{ij} + \delta]$, the vol. of each such closed rect. being $(2\delta)^n$, we can choose $\delta_\epsilon$ s.t $r \cdot (2\delta_\epsilon)^n < \epsilon$

2. Infact, a countably infinite set also has measure 0.

Lem : Let $A_1, A_2, \dots$ be contably infinite coll. of sets of measure 0. Then $A = \bigcup_i A_i$ also has measure 0.

Pf : for any $\epsilon > 0$, choose a cover for $A_i$ by closed rect. $V_{ij}$ s.t $\sum_j v(V_{ij}) < \epsilon/2^i$.

Then $\{V_{ij}\}_{i,j}$ is a cover of $A$ with $\sum_{i,j} v(V_{ij}) < \sum_i \epsilon/2^i = \epsilon$

Def$^n$ : Let $A \subseteq \mathbb{R}^n$. We say $A$ has content 0 if for $\epsilon > 0$, there are closed rect. $A_1, \dots, A_n$ covering $A$ s.t $\sum_i v(A_i) < \epsilon$

Lem : The set $[a,b] \subseteq \mathbb{R}$ does not have content 0.

Pf : We prove that for any finite coll. of intervals $[t_i, t_i']$, $i = 1, 2, \dots, n$ covering $[a,b]$, we should be $\sum (t_i' - t_i) \geqslant b - a$

Wlog, $[t_i, t_i'] \subseteq [a,b]$

Let $a = x_0 < x_1 < \dots < x_k = b$ be all the endpts. of the intervals $[t_i, t_i']$.

Then $[r_i, r_{i+1}]$ has to be a subset of $[t_i, t_i']$

Thus for each subinterval $[t_i, t_i']$ covering $[a,b]$, $t_i' - t_i = \sum\limits_{[r_i, r_{i+1}] \subseteq [t_i, t_i']} (r_{i+1} - r_i)$

Further $\sum\limits_i t_i' - t_i \geqslant \sum\limits_{i=1}^{l} r_{i+1} - r_i = b - a > 0$ since $a < b$

**Lem**: If $A$ is compact & has measure $0$, then it has content $0$.

**Pf**: If $A$ has measure $0$, then $A \subseteq \bigcup U_{i,\epsilon}$ with $\sum v(U_{i,\epsilon}) < \epsilon$

Since $A$ is compact, the cover $\{U_{i,\epsilon}\}$ has finite subcover $\{U_{i_1,\epsilon}, \dots, U_{i_n,\epsilon}\}$. Then $A$ has content $0$.

**Cor**: The set $[a,b]$ does not have measure $0$.

Content $0$ $\Rightarrow$ Measure $0$

Compact & Measure $0$ $\Rightarrow$ Content $0$

**Rem**: The analysis above can also be done in terms of open rectangles. Closed rect. are not necessary.

Recall $O(f,x)$. We have a b'nd fn $f: A \to \mathbb{R}$, $A \subseteq \mathbb{R}^n$

Let $x \in A$. define

$$M(f,\delta) = \sup\{f(y) : y \in A, \|y-x\| < \delta\}$$
$$m(f,\delta) = \inf\{f(y) : y \in A, \|y-x\| < \delta\}$$

$$\lim_{\delta \to 0} (M(f,\delta) - m(f,\delta)) = O(f,x)$$

Lem: Let $A \subseteq \mathbb{R}^n$ be a closed rect. & $f: A \to \mathbb{R}$ be a b'nd fn s.t $O(f,a) < \epsilon$ $\forall a \in A$ for a fixed $\epsilon > 0$. Then $\exists$ partition $P$ of $A$ s.t

$$U(f,P) - L(f,P) < \epsilon \cdot v(A)$$

Pf: Fix $x \in A$. Since $O(f,x) < \epsilon$, $\exists$ $\delta$-nbd of $x$ s.t

$$M(f,\delta) - m(f,\delta) < \epsilon$$

This means that in the $\delta$-nbd of $x$, the $\sup f(y)$ & $\inf f(y)$ are at dist. $< \epsilon$. Then there is an open rect. containing $U_x$ containing $x$ s.t $\sup_{y \in U_x} f(y) - \inf_{y \in U_x} f(y) < \epsilon$.

Similarly, we can find a closed rectangle $V_x$ s.t $x \subseteq V_x \subseteq U_x$

Since $A$ is compact, we have a finite cover $U_{x_1}, \dots, U_{x_n}$ of $A$.

Choose a partition $P$ s.t each $S \in P$ is contained in some $U_{x_i}$.

Then $M_S(f) - m_S(f) < \epsilon$ for each subrectangle $S$ of $P$.

Hence, $U(f,P) - L(f,P) = \sum_{S \in P} M_S(f) - m_S(f) < \epsilon \cdot v(A)$.

Thm: Let $A$ be a closed rect., $f : A \to \mathbb{R}$ be b'nd & let $B = \{x \in A : f \text{ is } \underline{\text{not}} \text{ cts. at } a\}$. Then $f$ is int'ble iff $B$ has measure 0.

Pf: Let us suppose that $B$ has measure 0.
If we define $B_\epsilon = \{x \in A : O(f, x) \geq \epsilon\}$ for $\epsilon > 0$.
Then $B_\epsilon \subseteq B$ & hence $B_\epsilon$ has measure 0. Then $B_\epsilon$ is compact & hence it has content 0.

Hence, $B_\epsilon$ has a finite cover by closed rect. $U_1, \dots, U_n$ s.t $\sum v(U_i) < \epsilon$.

Construct a partition $P$ s.t every $S \in P$ either has empty intersection with $B_\epsilon$ or is contained in one of the $U_i$.

For this partition $P$, $M(f, S) - m(f, S) < 2M$ where $M = \sup\limits_{x \in A} f(x)$ for each $S \in P$

For each $S \in P$ with $S \cap B_\epsilon = \emptyset$, $M(f, S) - m(f, S) < \epsilon$ & then one can refine the partition $P$ to obtain $P'$ s.t $U(f, P') - L(f, P') < \epsilon \cdot v(S)$ on $S$.
For the remaining subrectangles $(M(f, S) - m(f, S))\, v(S) < 2M \cdot v(S)$

for each $S \in P'$ with $S \cap B_\epsilon \neq \emptyset$, $\quad \sum v(S) \leq \sum v(U_i) < \epsilon$

Then $$U(f,P') - L(f,P') = \sum_{S \cap B_\epsilon = \emptyset} (M(f,S) - m(f,S))\, v(S) + \sum_{S \cap B_\epsilon \neq \emptyset} (M(f,S) - m(f,S))\, v(S)$$
$$< \epsilon \cdot v(A) + 2M \cdot \epsilon = (v(A) + 2M)\epsilon$$

Since this can we done for every $\epsilon > 0$, $f$ is int'ble.

Conversely, we assume that $f$ is int'ble. If we define $B_\epsilon$ as above, then $B = B_1 \cup B_{1/2} \cup B_{1/3} \ldots \cup B_{1/n} \ldots$

It suffice to show each $B_{1/n}$ has measure $0$.
Let $\epsilon > 0$ be fixed & choose a partition $P$ of $A$ with
$U(f,P) - L(f,P) < \epsilon/n$

Let $\mathcal{P}$ be the subset of $P$ of subrect. $S$ with $S \cap B_{1/n} \neq 0$.
Then $\mathcal{P}$ is a cover of $B_{1/n}$.

For each $S \in \mathcal{P}$, $\quad M(f,S) - m(f,S) \geq 1/n$

Then $$\frac{1}{n} \sum_{S \in \mathcal{P}} v(S) \leq \sum_{S \in \mathcal{P}} (M(f,S) - m(f,S))\, v(S)$$
$$\leq \sum_{S \in P} (M(f,S) - m(f,S))\, v(S)$$
$$= U(f,P) - L(f,P) < \epsilon/n$$

$\Rightarrow \sum_{S \in \mathcal{P}} v(S) < \epsilon$.

Since $\epsilon$ is arbitrary, $B_{1/n}$ has measure 0.

Def$^n$: Let $C$ be a b'nd subset of $\mathbb{R}^n$, $\underset{(\text{rectangle})}{C \subseteq A}$ & let $f : A \to \mathbb{R}$ be b'nd. Then $\int_C f = \int_A f \cdot \chi_C$ where $\chi_C$ is the characteristic fn$^n$ of $C$, whenever $\int_A f \cdot \chi_C$ makes sense.

Note: $\chi_C : \mathbb{R}^n \to \mathbb{R}$, $\chi_C(x) = \begin{cases} 1, & x \in C \\ 0, & x \notin C \end{cases}$

Recall, for $C \in \mathbb{R}^n$, the interior of $C$ is the largest open subset of $C$

Further, the interior of $\mathbb{R}^n \setminus C$ is called the exterior of $C$.

Finally, $\mathbb{R}^n = C^\circ \sqcup (\mathbb{R}^n \setminus C)^\circ \sqcup \partial C$, where $\partial C$ is the b'ry of $C$.

Pp$^n$: Let $C \subseteq \mathbb{R}^n$ be b'nd. The characteristic fn$^n$ $\chi_C$ is int'ble on any rect. containing $C$ iff $\partial C$ has measure zero.

Pf: If $x \in C^\circ$, then there is an open rect. $U$ containing $x$ inside $C$. T'fore, there is an open rect. $U$ containing $x$ s.t $\chi_C \equiv 1$. Then $\chi_C$ is cts. at $x$.

Sim. for $x$ in the exterior of $C$, $\chi_C \equiv 0$ in an open rect. containing $x$, hence $\chi_C$ is cts. at $x$.

Thus, all pts. $x \in \mathbb{R}^n$ where $\chi_C$ is <u>not</u> cts. are pts. in $\partial C$. Further if $y \in \partial C$, then every open set containing $y$ has $z \in C^\circ$ & $w \in (\mathbb{R}^n \setminus C)^\circ$. Hence, $\chi_C$ is <u>not</u> cts. at $y$.

Hence, $\partial C = \{x \in \mathbb{R}^n : \chi_C \text{ is } \underline{\text{not}} \text{ cts. at } x\}$.
The proof then follows.

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Let $A \in \mathbb{R}^n$, $B \subseteq \mathbb{R}^m$ & let $f : A \times B \to \mathbb{R}$ be a b'nd fx$^n$.
For $x \in A$, let $g_x$ denote the fx$^n$ on $B$, $g_x(y) = f(x, y)$
The supremum of lower sums & infimum of upper sums always exist for $g_x$. We define $L\int_B g_x$ to be the sup. of lower sums & $U\int_B g_x$ to be the inf. of upper sums.

These are called the lower & upper integrals of the given fx$^n$ on the given sets.

Thm: (Fubini)

Let $A \subseteq \mathbb{R}^n$, $B \subseteq \mathbb{R}^m$ be closed rect. & $f: A \times B \to \mathbb{R}$ be int'ble.

For $x \in A$, we define $g_x$ as above. Then if $L(x) = L\int_B g_x$ & $U(x) = U\int_B g_x$, we have that $L(x)$ & $U(x)$ are int'ble on $A$.

Further $\int_A L(x) = \int_A U(x) = \int_{A\times B} f$

Note: Fubini's thm only says that once $f$ is int'ble on $A \times B$, $\int_{A\times B} f$ is equal to $\int_A \left( \int_B f(x,y)\, dy \right) dx$.

These are called iterated integrals.

Pf: Let $P_A$, $P_B$ be partitions of $A$ & $B$ resp.

This gives a partition of $A \times B$ denoted by $P_{A\times B}$.

Each $S \in P_{A\times B}$ is of the form $S_A \times S_B$ where $S_A \in P_A$, $S_B \in P_B$.

Fix $x \in A$ & let $x \in S_A \in P_A$. For any $S_B \in P_B$

Note that $g_x$ is defined on $B$, so on $S_B$ & $f$ is defined on $S_A \times S_B$.

$$m_{f(S_A\times S_B)} \leq m_{\underbrace{g_x(S_B)}_{f(\{x\}\times S_B)}} \quad \text{for any } S_B.$$

$$L(x) = \lim_{P_B} \sum_{S_B \in P_B} m_{g_x(S_B)} \cdot v(S_B)$$

for the partition $P_{A \times B}$,

$$L(f, P_{A \times B}) = \sum_{\substack{S_A \in P_A \\ S_B \in P_B}} m_{f(S_A \times S_B)} \cdot v(S_A \times S_B)$$

$$= \sum_{S_A \in P_A} \left( \sum_{S_B \in P_B} m_{f(S_A \times S_B)} \cdot v(S_B) \right) v(S_A)$$

$$\leq \sum_{S_A \in P_A} \left( \sum_{S_B \in P_B} m_{g_x(S_B)} \cdot v(S_B) \right) v(S_A)$$

$$\leq \sum_{S_A \in P_A} \left( L(g_x, P_B) \right) v(S_A)$$

$$\leq \sum_{S_A \in P_A} \left( L\int_{S_A} L(x) \right) v(S_A)$$

$$= L(h_x, P_A)$$

So, $L(f, P_{A \times B}) \leq L(h_x, P_A) \leq U(U_x, P_A) \leq U(f, P_{A \times B})$

Let $A$ be a rect. in $\mathbb{R}^n$ & $f : A \to \mathbb{R}$ be a bounded fn.

For any partition $P$ of $A$, we form $L(f,P)$ & $U(f,P)$

$$L(f,P) = \sum_{S\in P} m_S(f)\, v(S) \;, \qquad U(P,f) = \sum_{S\in P} M_S(f)\, v(S).$$

We know that for any two partitions $P$ & $Q$, $L(f,P) \leq U(f,Q)$.

T'fore $\sup_P L(f,P)$ & $\inf_P U(f,P)$ exist.

We call $\sup_P L(f,P)$ as the lower integral of $f$ on $A$, denote it by $L\int_A f$. Sim. $\inf_P U(f,P)$ is called the upper integral of $f$ on $A$ & it is denoted by $U\int_A f$.

$$L(f,P) \leq L\int_A f \leq U\int_A f \leq U(f,Q) \quad \text{for any two partitions}$$

Let $A \subseteq \mathbb{R}^n$, $B \subseteq \mathbb{R}^m$ & let $f : A\times B \to \mathbb{R}$ be a b'nd fn

Let $P_A$ be a partition of $A$ & $P_B$ be a partition of $B$.

Then $P_A \times P_B$ is a partition of $A\times B$. Each element of $P_A\times P_B$ is of the form $S_A \times S_B$ where $S_A \in P_A$, $S_B \in P_B$.

Now, for $x \in A$, define $g_x : B \to \mathbb{R}$ by $g_x(y) = f(x,y)$.

Then we define $\mathcal{L}(x) = L\int_B g_x$, $\mathcal{U}(x) = U\int_B g_x$

Fubini's thm : Let the not$^n$s be as above. Further assume that $f$ is int'ble on $A\times B$. Then both $\mathcal{L}$ & $\mathcal{U}$ are int'ble on $A$ & $\int_A \mathcal{L} = \int_A \mathcal{U} = \int_{A\times B} f$

$$\int_{A\times B} f = \int_A \mathcal{L} = \int_A \left( L\int_B g_x \right) \quad , \quad \int_{A\times B} f = \int_A \mathcal{U} = \int_A \left( U\int_B g_x \right)$$

Pf: Suppose $x \in S_A$, then $m_{S_A\times S_B}(f) \leq m_{S_B}(g_x)$ for any $S_B \in P_B$.

Then $\sum_{S_B\in P_B} m_{S_A\times S_B}(f)\, v(S_B) \leq \sum_{S_B\in P_B} m_{S_B}(g_x)\, v(S_B) = L(g_x, P) \leq \mathcal{L}(x)$

Further, $\sum_{S_B\in P_B} m_{S_A\times S_B}(f) \cdot v(S_B) \leq m_{S_A}(\mathcal{L})$

$$\sum_{S_A\in P_A} \left( \sum_{S_B\in P_B} m_{S_A\times S_B}(f) \cdot v(S_B) \right) v(S_A) \leq \underbrace{\sum_{S_A\in P_A} m_{S_A}(\mathcal{L})\, v(S_A)} = L(\mathcal{L}, P_A)$$

we get $L(f, P_{A\times B}) \leq L(\mathcal{L}, P_A)$

$\leq U(\mathcal{L}, P_A)$

$\leq U(\mathcal{U}, P_A) \leq U(f, P_{A\times B})$

$$\sum_{S_A\times S_B\in P_{A\times B}} m_{S_A\times S_B}(f)\, v(S_A\times S_B) = L(f, P_{A\times B})$$

Since $\sup L(f, P_{A\times B}) = \inf U(f, P_{A\times B})$, we get $\sup L(\mathcal{L}, P_A) = \inf U(\mathcal{L}, P_A)$

Then $\mathcal{L}$ is int'ble on A & $\int_A \mathcal{L} = \int_{A\times B} f$.

A similar argument shows that $\int_A \mathcal{U} = \int_{A\times B} f$

In fact, we have a similar result

$$\int_{A\times B} f = \int_B \left( L\int_A f(x,y)\,dx \right) dy = \int_B \left( U\int_A f(x,y)\,dx \right) dy$$

eg: $$\int_{[0,1]\times[0,1]} \left( \frac{x-y}{(x+y)^3}\,dx \right) dy = \frac{1}{2} \quad \& \quad \int_{[0,1]\times[0,1]} \left( \frac{x-y}{(x+y)^3}\,dy \right) dx = \frac{-1}{2}$$

What is going wrong here?

The fx^n is unbounded (blows up as $(x,y)\to(0,0)$)

Can we modify the behaviour of f on a set of measure 0 to get integrability? No, since the values of iterated integrals are diffrent.

Recall, for any b'nd subset $C$ in $\mathbb{R}^n$ & a b'nd fcn $f: C \to \mathbb{R}$, we defined $\int_C f = \int_A f \cdot \chi_C$ where $C \subseteq A$ rect. & $f$ is defined on $A \setminus C$ in any way if $\int_A f \cdot \chi_C$ makes sense.

$\int_C 1$ exists iff $\partial C$ has measure $0$.
This value is called the n-dimensional vol. of $C$.

$\int_{[a,b]} 1 = b-a$ if $[a,b]$ is considered as a 1-dim object.

but $\int_{[a,b]} 1 = 0$ if $[a,b]$ is treated as an n-dim object for $n > 1$

# Partition of Unity

**Recall**: 1. If $C$ is a compact subset of $\mathbb{R}^n$ & $U$ is an open subset of $\mathbb{R}^n$ with $C \subseteq U$, then $\exists$ a compact $D$ with $C \subseteq D^{\circ} \subseteq D \subseteq U$

2. If $A$ is compact subset of $\mathbb{R}^n$ & $A \subseteq U$, open in $\mathbb{R}^n$, then $\exists$ a $C^{\infty}$ fxn $f$ defined on some open subset containing $A$ which is positive on $A$ & vanishes outside a closed subset contained in $U$

Given any $f : \mathbb{R}^n \to \mathbb{R}$, $\operatorname{supp}(f) = \overline{\{x \in \mathbb{R}^n : f(x) \neq 0\}}$

The second ppt. can be strengthened by getting the fxn $f$ to be constant 1 on $A$.

Thm: Let $A \subseteq \mathbb{R}^n$. Then there is a coll. $\Phi$ of real-valued $C^\infty$-fxn's defined on some open subset containing $A$ s.t.

1. $\forall\, x \in A$, $0 \leq |\varphi(x)| \leq 1$ $\forall\, \varphi \in \Phi$
2. $\forall\, x \in A$, $\exists$ open set $V_x$ containing $x$ s.t only finitely many $\varphi \in \Phi$ are non-zero on $V_x$
3. $\forall\, x \in A$, $\sum_{\varphi \in \Phi} \varphi(x) = 1$.
4. If $\mathcal{O}$ is an open cover of $A$, then $\Phi$ can be chosen s.t for every $\varphi \in \Phi$, $\text{supp}(\varphi)$ is contained in some $U \in \mathcal{O}$

The family $\Phi$, with the first 3 ppts. above, is called a partition of unity on A. Further, with the 4th ppt., one says that the partition of unity is 'subordinate to the open cover $\mathcal{O}$'.

Pf: Case 1 : A is compact. Then $\mathcal{O}$ has a finite subcover $\{U_1, U_2, \dots, U_n\}$ for A. We will construct a partition of unity on A subordinate to this subcover.

Define $C_1 = A \setminus (U_2 \cup \dots \cup U_n)$. This is a compact subset of A. Infact $C_1 \subseteq U_1$.

There exists a compact $D_1$ s.t $C_1 \subseteq D_1^o \subseteq D_1 \subseteq U_1$

Observe, $\{D_1^o, U_2, \ldots, U_n\}$ is also an open cover of A.

Now, define $C_2 = A \setminus (D_1^o \cup U_3 \cdots \cup U_n)$. This is a compact subset of $U_2$

We get a compact $D_2$ with $C_2 \subseteq D_2^o \subseteq D_2 \subseteq U_2$.

Continuing this way, we get compact subsets $D_1, \ldots, D_n$ s.t

$\{D_1^o, \ldots, D_n^o\}$ is an open cover of A with $D_i^o \subseteq U_i$.

Given a $D_i$, there is a $C^\infty$ fx$^n$ $\psi_i$ defined on some open set containing $D_i$ with $\text{supp}(\psi_i) \subseteq U_i$ & $\psi_i$ positive on $D_i$.

There is an open subset $U$ containing A s.t

$\psi_1(x) + \psi_2(x) + \ldots + \psi_n(x) > 0$ on $U$.

Now, define $\varphi_i(x)$ on $U$ by $\varphi_i(x) = \dfrac{\psi_i(x)}{\psi_1(x) + \ldots + \psi_n(x)}$

Now, $|\varphi_i(x)| \leq 1 \quad \forall x \in U$.

If f is a $C^\infty$ fx$^n$ which is 1 on A & $\text{supp}(f) \subseteq U$.

Then $\{f \cdot \varphi_1, \ldots, f \cdot \varphi_n\}$ is required partition of unity on A subordinate to the cover $\{D_1^o, \ldots, D_n^o\}$, hence subordinate to $\{U_1, \ldots, U_n\}$.

Case 2: $A = A_1 \cup A_2 \cup \dots$ where each $A_i$ is a compact subset of $A_{i+1}^{o}$.

Consider the open cover $\mathcal{O}$ of $A$. Fix an $i$.

For each $U \in \mathcal{O}$, consider $U \cap (A_{i+1}^{o} \setminus A_{i-2})$.

This gives an open cover of $A_{i+1}^{o} \setminus A_{i-2}$, in particular, we get an open cover $\mathcal{O}_i$ of $A_i \setminus A_{i-1}^{o}$ which is a compact subset of $A$.

The previous case says that we have a partition of unity on $A_i \setminus A_{i-1}^{o}$ subordinate to $\mathcal{O}_i$

Now, if $x \in U$ is in $A_i^{o}$, then the fn's consisting of the constructed partition of unity on $A_{i+1}$ vanish on $x$.

Thus, taking the union of these partitions of unity gives us the required partition of unity on $A$.

Case 3: $A$ is open

Define $A_i = \{ x \in \mathbb{R}^n : \|x\| \leq i \ , \ d(x, \partial A) \geq \frac{1}{i} \}$

Note, each $A_i$ is closed & bounded subset of $\mathbb{R}^n$, hence it is a compact subset of $\mathbb{R}^n$. Further $A_i \subseteq A_{i+1}^{o}$.

$$\|x\| \leq i \Rightarrow \|x\| < i+1 \quad \& \quad d(x, \partial A) \geq \frac{1}{i} \Rightarrow d(x, \partial A) > \frac{1}{i+1}$$

$$A_i \subseteq \underset{\text{(open)}}{\{ x \in \mathbb{R}^n : \|x\| < i+1 \ , \ d(x, \partial A) > \frac{1}{i+1} \}} \subseteq A_{i+1}$$

$\Rightarrow A_i \subseteq A_{i+1}^{\circ}$

Use case 2.

**Case 4:** General $A$

Let $\mathcal{O}$ be an open cover of $A$ & define $\mathcal{U} = \bigcup_{U \in \mathcal{O}} U$.

Previous case gives a partition of unity for $\mathcal{U}$ subordinate to $\mathcal{O}$.

It follows that it is also a partition of unity for $A$ subordinate to $\mathcal{O}$.

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We say that an open cover $\mathcal{O}$ of an open set $\mathcal{U}$ is 'admissible' if each member of $\mathcal{O}$ is a subset of $\mathcal{U}$.

**Rem:**

1. Let $U \subseteq \mathbb{R}^n$ open, bounded, $f: U \to \mathbb{R}$ is bounded & assume that the set of discontinuities of $f$ has measure zero. Then we know $\int_{\mathcal{U}} f$ exists. We also have that $\int \varphi \cdot |f|$ exists for each $\varphi \in \Phi$, a partition of unity for $U$ subordinate to an admissible open cover $\mathcal{O}$ of $U$

2. We say that $\int_{\mathcal{U}} f$ exists (in the general sense) if $\sum_{\varphi \in \Phi} \int_{U} \varphi \cdot |f|$ exists. The above discussion says that when the set of discontinuities of $f$ has measure zero, $\int_{\mathcal{U}} f$ exists in the general sense.

3. When $\int_U f$ exists in the general sense, then $\sum_{\varphi \in \Phi} \left| \int \varphi \cdot f \right|$ exists. This means that the series $\sum_{\varphi \in \Phi} \int_U \varphi \cdot f$ conv. absolutely, and hence, is itself conv.

4. The existence of $\int_U f$ in the general sense does not depend on the choice of $\Phi$ or on the admissible open cover $\mathcal{O}$.

Note : 1. Spivak considers a partition of unity to always have compact support.

More precisely, every $\varphi \in \Phi$ has a compact set $C_\varphi$ outside of which $\varphi$ is zero.

It is always possible to get such a partition of unity

2. If $C$ is a compact set, then only finitely many $\varphi \in \Phi$ are non-zero

Let $A \subseteq \mathbb{R}^n$ & let $f : A \to \mathbb{R}$ be a fn^n that is locally b'nd.

That is, for any $x \in A$, $\exists$ an open $U_x$ s.t $f$ is b'nd on $U_x$.

Now, assume that the set of discontinuities of $f$ has measure zero. Then for each $\varphi \in \Phi$, $\int_A \varphi \cdot |f|$ exists.

We say that f is int'ble in the general sense if

$\sum_{\varphi\in\Phi} \int_A \varphi\cdot|f|$ conv.

<u>Rem</u>: If f is int'ble in the general sense, then $\sum_{\varphi\in\Phi} |\varphi\cdot f|$ conv.

This means that $\sum_{\varphi\in\Phi}\int_A \varphi\cdot f$ conv. abs., hence it conv.

We then define $\int_A f = \sum_{\varphi\in\Phi}\int_A \varphi\cdot f$

<u>Thm</u>: Let $\mathcal{O}$ be an open admissible cover of A, $\Phi$ a partition of unity for A subordinate to $\mathcal{O}$ & let $f: A\to\mathbb{R}$ be locally b'nd. Assume that f is int'ble over A in the general sense.

For any admissible open cover $\mathcal{O}'$ of A & any partition of unity $\Psi$ for A subordinate to $\mathcal{O}'$, $\sum_{\psi\in\Psi}\int_A \psi\cdot|f|$ conv.

Further, $\sum_{\varphi\in\Phi}\int_A \varphi\cdot|f| = \sum_{\psi\in\Psi}\int_A \psi\cdot|f|$

<u>Pf</u>: Let $\varphi\in\Phi$, then the support of $\varphi$ is a compact subset of $\mathbb{R}^n$, call it C. Only finitely many $\psi\in\Psi$ are non-zero on C. Furthermore, $\sum_{\psi\in\Psi}\psi = 1$ on C.

$$\Rightarrow \sum_{\psi\in\Psi}\psi\cdot\varphi = \varphi$$

$$\Rightarrow \int_A \varphi\cdot f = \sum_{\psi\in\Psi}\int_A \psi\cdot\varphi\cdot f$$

We then get, $\sum_{\varphi\in\Phi} \int_A \varphi\cdot f = \sum_{\varphi\in\Phi}\sum_{\psi\in\Psi} \int_A \psi\cdot\varphi\cdot f$

In particular, replacing $f$ with $|f|$, we get conv. of $\sum_{\varphi\in\Phi}\sum_{\psi\in\Psi}\int_A \varphi\cdot\psi\cdot|f|$

It implies that $\sum_{\varphi\in\Phi}\sum_{\psi\in\Psi}\int_A \psi\cdot\varphi\cdot f$ is absolutely conv.

Thus, $$\sum_{\varphi\in\Phi}\left(\sum_{\psi\in\Psi}\int_A \psi\cdot\varphi\cdot f\right) = \sum_{\psi\in\Psi}\left(\sum_{\varphi\in\Phi}\int_A \psi\cdot\varphi\cdot f\right)$$

$$\Rightarrow \sum_{\varphi\in\Phi}\int_A \varphi\cdot f = \sum_{\psi\in\Psi}\int_A \psi\cdot f$$

Thm: If $A$ is a b'nd set & $f$ is also b'nd, disc($f$) has measure zero. Then $f$ is int'ble in the general sense.

Pf: Let $|f|<M$. Since $A$ is b'nd, for any given partition of unity $\Phi$ for $A$, only finitely many are non-zero on $A$. Call that set $F$.

$$\sum_{\varphi\in\Phi}\int_A \varphi\cdot|f| = \sum_{\varphi\in F}\int_A \varphi\cdot|f| \leqslant \sum_{\varphi\in F} M\int_A \varphi = M\int_A \sum_{\varphi\in F}\varphi$$

$$= M\int_A 1 = M\cdot v(A)$$

Ex: If $A$ is b'nd & Jordan m'ble, then for any $\epsilon > 0$, $\exists$ a compact set $C_\epsilon \subseteq A$ which is also Jordan m'ble s.t $\int_{A \setminus C_\epsilon} 1 \leq \epsilon$

## Change of variables

Thm: Let $A \subseteq \mathbb{R}^n$ be an open set & $g : A \to \mathbb{R}^n$ be a one-one, ctsly. diff fn with $\det g'$ non-vanishing for every $x \in A$.

Then $\int_{g(A)} f = \int_A (f \circ g)(\det g')$ for any int'ble $f : g(A) \to \mathbb{R}$

Sard's thm: Let $A \subseteq \mathbb{R}^n$ be open & $g : A \to \mathbb{R}^n$ be ctsly. diff. Let $B = \{x \in A : \det g'(x) = 0\}$. Then the set $g(B)$ has measure $0$.

The set where $g'$ has det zero is called the critical set for $g$.

Pf: Choose a closed rect. $U$, whose sides have length $l$, inside $A$.

By the def^n of $Dg$, for any $x \in U$, $\lim_{h \to 0} \frac{\| Dg(x)(h) - (g(x) - g(x+h)) \|}{\|h\|} = 0$

Fix $\epsilon > 0$. $\exists N$ s.t for any $x \in U$, $\| Dg(x) - (g(x) - g(y)) \| < \epsilon \|x - y\|$ whenever $x$ & $y$ are in a subrect. of $U$ whose sides are equal to $l/N$.

Choose any such subcube $S$ of side $l/N$.

If $x \in S \cap B$, then $Dg(x) \in \begin{pmatrix} \text{an } n-1 \text{ dim subsp. } V \\ \text{of } \mathbb{R}^n \end{pmatrix}$

Further, the above inequality is

$$\|Dg(x) - (g(x) - g(y))\| < \epsilon \|x-y\| \leq \epsilon \sqrt{n}\, l/N$$

T'fore $g(x) - g(y)$ lies in the thickening of the vector sp. $V$ by $\epsilon \sqrt{n} \frac{l}{N}$

More precisely, $g(y)$ is in the $\epsilon \sqrt{n} \frac{l}{N}$ nbd of $V + g(x)$.

This holds $\forall\, y \in S$.

By lemma 2.10, within $V$, $\|g(x) - g(y)\| < \epsilon M \|x - y\|$

$$\leq \epsilon M \sqrt{n} \cdot \frac{l}{N}$$

Thus, $g(y)$ is in a cylinder whose base is of radius $\epsilon M \sqrt{n} \frac{l}{N}$
(n+1 sphere)

& is of height $2\epsilon \sqrt{n} \frac{l}{N}$

The vol. of this cylinder is $C\, \epsilon^{n-1} M^{n-1} (\sqrt{n})^{n-1} \left(\frac{l}{N}\right)^{n-1} \times 2\epsilon \sqrt{n} \frac{l}{N}$

$$= C_1 \left(\frac{l}{N}\right)^n \epsilon^n M^{n-1}$$

$g(S)$ is contained in a cylinder of volume $C_1\left(\frac{l}{N}\right)^n \epsilon^n M^{n-1} \leq C_2\left(\frac{l}{N}\right)^n \epsilon^n$

The no. of all subrect. in $U$ is $N^n$. The sets $g(S)$ where $S \cap B$ is non-empty are all contained in a set of vol. $\leq C_2 l^n \epsilon^n$

This is true for every $\epsilon > 0$. Thus $g(U \cap B)$ has measure $0$.
Now, $A$, being an open subset of $\mathbb{R}^n$, can be covered with countably many such rect. $U$, with $g(U \cap B)$ of measure $0$.

Hence $g(A \cap B) = g(B)$ is of measure zero.