MA412

Complex Analysis

Defining complex nos. $\mathbb{C}$

Start with the complex field $\mathbb{R}$.

Consider $\mathbb{R} \times \mathbb{R}$ & define $+$ & $\cdot$ on $\mathbb{R} \times \mathbb{R}$.

$$(a,b) + (c,d) = (a+b, c+d)$$

$$(a,b) \cdot (c,d) = (ac-bd, ad+bc)$$

$$\mathbb{C} := (\mathbb{R} \times \mathbb{R}, +, \cdot)$$

Check: Commutativity: $z + w = w + z$

$z \cdot w = w \cdot z$

Associativity: $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$

$(z_1 \cdot z_2) \cdot z_3 = z_1 \cdot (z_2 \cdot z_3)$

Distributivity: $z_1 \cdot (z_2 + z_3) = z_1 \cdot z_2 + z_1 \cdot z_3$

Additive identity: $(0,0)$

Multiplicative identity: $(1,0)$

Embed $\mathbb{R} \hookrightarrow \mathbb{C}$

$x \mapsto (x, 0)$

and we define, $i = (0, 1)$

$$(0,1)\cdot(0,1) = (0\cdot 0 - 1\cdot 1,\ 0\cdot 1 + 1\cdot 0)$$
$$= (-1, 0)$$

So, $i^2 = -1$

Write $(a, b) = a + ib$, $a, b \in \mathbb{R}$

$(a+ib)(a-ib) = a^2 + b^2$

If $z = a + ib \in \mathbb{C}$.

Def. $\bar{z} = a - ib$ & $|z|^2 = z \cdot \bar{z}$

1.1 helps define a metric topology on $\mathbb{C}$.

$a = Re(z)$

$b = Im(z)$

Check:

1. $\overline{(z+w)} = \bar{z} + \bar{w}$

2. $\overline{z \cdot w} = \bar{z} \cdot \bar{w}$

Also, $z \cdot w = z \cdot w$

$\Rightarrow z w \, \overline{z \cdot w} = z \cdot w \cdot \bar{z} \cdot \bar{w} = z \cdot \bar{z} \cdot w \cdot \bar{w}$

$\Rightarrow |zw|^2 = |z|^2 \cdot |w|^2$

$\Rightarrow |zw| = |z| \cdot |w|$

Note: $|\cdot|$ satisfies the Δ inequality i.e

$|z_1 - z_2| \leq |z_1 - z_3| + |z_3 - z_2|$

Suffices to show that, $|z_1 + z_2| \leq |z_1| + |z_2|$

Pf: $|z_1 + z_2|^2 = (z_1 + z_2)(\bar{z}_1 + \bar{z}_2)$

$= \underbrace{z_1 \bar{z}_1}_{|z_1|^2} + \underbrace{z_2 \bar{z}_2}_{|z_2|^2} + \underbrace{z_1 \bar{z}_2 + \bar{z}_1 z_2}_{(z_1\bar{z}_2 + \overline{z_1\bar{z}_2}) = 2\mathrm{Re}(z_1\bar{z}_2)}$

$\leq 2|z_1 \bar{z}_2|$

$\leq |z_1|^2 + |z_1|^2 + 2|z_1 \bar{z}_2|$ $\left(\because |\mathrm{Re}(z)| \leq |z| \ \& \ |\mathrm{Im}(z)| \leq |z| \right)$

$\leq |z_1|^2 + |z_1|^2 + 2|z_1||\bar{z}_2|$

$\leq |z_1|^2 + |z_1|^2 + 2\underbrace{|z_1||z_2|}_{|z_1 z_2|}$ $\left(\because |\bar{z}|^2 = \bar{z}\bar{\bar{z}} = \bar{z} \cdot z = |z|^2 \right)$

$\leq (|z_1| + |z_2|)^2$

$\Rightarrow |z_1 + z_2| \leq |z_1| + |z_2|$

Similarly, $|z_1 + \dots + z_n| \leq |z_1| + \dots + |z_n|$

Pf. By ind^n

Say $|z_1 + \dots + z_{n-1}| \leq |z_1| + \dots + |z_{n-1}|$

Then, $|z_1 + \dots + z_n| \leq |z_1 + \dots + z_{n-1}| + |z_n|$

$$\leq |z_1| + \dots + |z_n|$$

Shown

1. $|z_1 + z_2| \leq |z_1| + |z_2|$
2. $|z_1 \cdot z_2| = |z_1||z_2|$
3. $|z| = 0 \Leftrightarrow z = 0$

This makes $|.|$ into an Archimedean norm.

Cauchy-Schwarz inequality:

$$\left|\sum_{k=1}^{n} z_k w_k\right|^2 \leq \left(\sum_{k=1}^{n} |z_k|^2\right)\left(\sum_{k=1}^{n} |w_k|^2\right)$$

<u>Pf</u>. Consider $\sum_{k=1}^{n} |z_k - \lambda \bar{w}_k|^2 \geq 0$ for some $\lambda \in \mathbb{C}$

Choose λ in a clever manner.

<u>Topology on $\mathbb{C}$</u>

The topology generated by

$$B(z, r) = \{ w : |z - w| < r \}$$

<u>Ppt</u>: $(\mathbb{C}, |.|)$ is complete

Given a Cauchy seq. $\{z_n\}_{n \geq 1}$, we want to show it converges in $\mathbb{C}$.

i.e given $\epsilon > 0$, $\exists N$ s.t $\forall m, n \geq N$, $|z_n - z_m| \leq \epsilon$

Recall, $Re(w), Im(w) \leq |w|$

$\Rightarrow |Re(z_n - z_m)| < \epsilon$ & $|Im(z_n - z_m)| < \epsilon$ for $n, m \geq N$

$\Rightarrow |Re(z_n) - Re(z_m)| < \epsilon$ & $|Im(z_n) - Im(z_m)| < \epsilon$

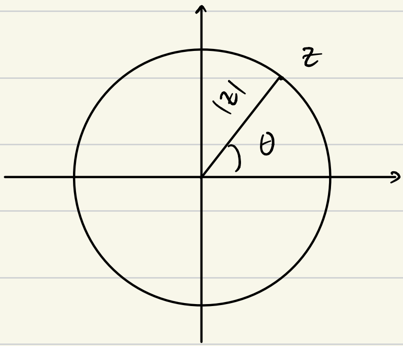
$\Rightarrow \{Re(z_n)\}_{n \geq 1}$ & $\{Im(z_n)\}_{n \geq 1}$ are Cauchy seq.

$\Rightarrow \exists\, a, b \in \mathbb{R}$, s.t $|a - Re(z_n)| < \epsilon/2$ & $|b - Im(z_n)| < \epsilon/2$

$$\Rightarrow |(a+ib) - z_n| = |(a - Re(z_n)) + (b - Im(z_n))i|$$
$$\leq \underbrace{|a - Re(z_n)|}_{\leq \epsilon/2} + \underbrace{|b - Im(z_n)|}_{\leq \epsilon/2}\, \underbrace{|i|}_{1}$$
$$\leq \epsilon$$

Polar coordinates

$\theta \in [0, 2\pi)$



$z = a + ib$ can also be stated in terms of $|z|$ & θ

$$z = |z| \cdot (\cos(\theta) + i \sin(\theta))$$

But this requires rigorously defining $\sin\theta$ & $\cos\theta$.

Recall: Given a series $\sum_{n=0}^{\infty} a_n$, it is said to converge absolutely if $\sum_{n=0}^{\infty} |a_n| < \infty$

Absolute conv. $\Rightarrow$ conv.

i.e $\sum_{n\geq 0} |a_n| < \infty$, that is given $\epsilon > 0$, $\exists N$ s.t $\sum_{n>N} |a_n| < \epsilon$

Consider $z_n = \sum_{k=1}^{n} a_k$

Goal: Show that $\{z_n\}_{n\geq 0}$ is a Cauchy seq.

Given $\epsilon > 0$, choose some N. Then $n > m > N$,

$$|z_n - z_m| = \left|\sum_{k=m+1}^{n} a_n\right| \leq \sum_{k=m+1}^{n} |a_k| < \epsilon$$

Since $\mathbb{C}$ is complete, $\{z_n\}_{n\geq 1}$ conv.

Geometric series

This converges absolutely when $|z|<1$.

Diverges when $|z|>1$

The series converges uniformly on $B(0,r)$ & $r<1$.

Given $\sum_{n=0}^{\infty} a_n(z-c)^n$.

Def. R as $\frac{1}{R} = \limsup |a_n|^{1/n}$ if finite

$R=0$ if $\limsup |a_n|^{1/n} = \infty$

$R=\infty$ if $\limsup |a_n|^{1/n} = 0$

Then the series conv. inside $|z-c|<R$

Diverges when $|z-c|>R$ & converges uniformly on $|z-c|\leq r$ for $0<r<R$ (when $R>0$)

We will study $\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$

This converges on $\mathbb{C}$.

$$\cos\theta = \frac{\exp(i\theta) + \exp(-i\theta)}{2}$$

$$\sin\theta = \frac{\exp(i\theta) - \exp(-i\theta)}{2i}$$

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}, \quad \text{when } |z| < 1$$

$$\sum_{n=0}^{N} z^n + \underbrace{\sum_{n=N+1}^{\infty} z^n}_{\text{Error}}$$

$$(1-z)\left(\sum_{n=0}^{N} z^n\right) = \sum_{n=0}^{N} z^n - \sum_{n=1}^{N+1} z^n$$

$$= 1 - z^{N+1}$$

$$\Rightarrow \sum_{n=0}^{N} z^n = \frac{1-z^{N+1}}{1-z}$$

$$\Rightarrow \lim_{N\to\infty} \sum_{n=0}^{N} z^n = \lim_{N\to\infty} \frac{1-z^{N+1}}{1-z} = \frac{1}{1-z}, \text{ if } |z|<1$$

since when $|z|<1$, $\lim_{N\to\infty} |z|^N = 0$ & $\lim_{N\to\infty} z^{N+1}$ exists

Sim., $\sum_{n=0}^{\infty} (z-c)^n$ converges for $|z-c|<1$

c ↳ centre of the power series

More generally, consider $\sum_{n=0}^{\infty} a_n (z-c)^n$

Def. $\frac{1}{R} = \limsup |a_n|^{1/n}$ for $0 < \limsup |a_n|^{1/n} < \infty$

$R = 0$ for $\limsup |a_n|^{1/n} = \infty$

$R = \infty$ for $\limsup |a_n|^{1/n} = 0$

Ppⁿ: Let $R>0$

1. If $|z-c|<R$, then the series conv. abs.

2. If $|z-c|>R$, the series diverges.

3. On $\Omega \subseteq B_c(R)$, the series conv. uniformly.
compact

Pf: Let $c=0$.

$$\frac{1}{R} = \limsup |a_n|^{1/n}$$

Let $z \in B_0(R)$

$\exists\, r>0$ s.t $|z|<r<R \Rightarrow \frac{1}{r} > \frac{1}{R}$

$\exists\, N$ s.t $n \geq N$, then $\frac{1}{r} > |a_n|^{1/n}$

$$\Rightarrow \frac{1}{r^n} > |a_n|$$

$$\Rightarrow \frac{|z|^n}{r^n} > |a_n||z|^n$$

$$\Rightarrow \underbrace{\sum_{n=N}^{\infty} \frac{|z|^n}{r^n}}_{|z|<r \;\Rightarrow\; LHS < \infty} > \sum_{n=N}^{\infty} |a_n z^n|$$

$$\Rightarrow \sum_{n=N}^{\infty} |a_n z^n| < \infty$$

$$\Rightarrow \sum_{n=0}^{\infty} |a_n z^n| = \sum_{n=0}^{N-1} |a_n z^n| + \sum_{n=N}^{\infty} |a_n z^n| < \infty$$

<u>Weierstrass M-test</u> : Let open $\Omega \subseteq \mathbb{C}$ & $\{u_n(z)\}_{n\geq 1}$ be a seq. of fn^n s.t $|u_n(z)| \leq M_n$ $\forall\, z \in \Omega$ & $\sum M_n < \infty$. Then $\sum_{n\geq 1} u_n(z)$ converges uniformly on Ω.

Pf. 3. Choose $r < R$ s.t $\overline{B_0(r)} \supseteq \Omega$

(Take $\{|z| : z \in \Omega\}$, choose $r = \sup\{|z| : z \in \Omega\}$)

We will show uniform conv. on $\overline{B(0,r)}$

for that, find ρ s.t $r < \rho < R$.

Apply W-M test to $\sum_{n=0}^{\infty} a_n z^n$ for $z \in \overline{B(0,r)}$

Find M_n s.t $|a_n z^n| < M_n$ & $\sum_{n=1}^{\infty} M_n < \infty$

$|z| \leq r \Rightarrow |z|^n \leq r^n$

$\frac{1}{R} = \limsup |a_n|^{1/n}$ & $\rho < R$

$\Rightarrow \exists N$ s.t $|a_n| < \frac{1}{\rho^n} \quad \forall\, n \geq N.$

Take $M = \left(\frac{r}{\rho}\right)^n$ for $n \geq N$.

$\Rightarrow$ 1. $|a_n z^n| \leq M_n$

$M_n = |a_n r^n|$ for $0 \leq n < N$

$\Rightarrow$ 2. $\sum_{n \geq 1} M_n < \infty$

Pf. of W-M test:

Consider $f_n(z) = \sum_{k=1}^{n} u_k(z)$ & we want to show that $\{f_n(z)\}$ conv. uniformly.

We will show that for a fixed z, $\lim_{n\to\infty} f_n(z)$ exists by showing $\{f_n(z)\}_{n\geq 1}$ is a Cauchy seq. $\forall z \in \Omega$.

Given $\epsilon > 0$, consider $|f_n(z) - f_m(z)|$ for $n > m$

$$|f_n(z) - f_m(z)| \leq \sum_{k=m+1}^{n} |u_k(z)| \leq \sum_{k=m+1}^{n} M_k$$

Since $\sum_{k=1}^{\infty} M_k$ conv., we have $\sum_{k=N}^{\infty} M_k < \epsilon$ for some N.

So, for $n > m > N$, $|f_n(z) - f_m(z)| < \epsilon$

Def. $f : \Omega \to \mathbb{C}$ by

$$f(z) = \lim_{n\to\infty} f_n(z)$$

For uniform conv., $|f(z) - f_n(z)| = \left| \sum_{k=n+1}^{\infty} u_k(z) \right|$

$$\leq \sum_{k=n+1}^{\infty} |u_k(z)| < \sum_{k=n+1}^{\infty} M_k$$

$\Rightarrow \exists\ N$ indep. of z s.t $\sum_{k=n+1}^{\infty} |u_k(z)| < \epsilon$ when $n \geq N$

So, conv. is uniform.

Pf. 2. If $|z| > R$.

Choose $r > R$ s.t $|z| > r > R$

$$\Rightarrow \frac{1}{r} < \frac{1}{R} = \limsup |a_n|^{1/n}$$

There are ∞-many n s.t $\frac{1}{r} < |a_n|^{1/n} \Rightarrow \frac{1}{r^n} < |a_n|$

$$\Rightarrow \underbrace{\frac{|z|^n}{r^n}}_{>1} < |a_n||z|^n$$

$\Rightarrow |a_n z^n| > 1$ for ∞-many n.

$\Rightarrow \sum a_n z^n$ diverges

Plan

Approach

1. Def $\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$

2. Justify $e^z = \exp(z)$ makes sense

3. Def. $\cos(z)$ & $\sin(z)$ in terms of $\exp(z)$.

What does z^w mean for $z, w \in \mathbb{C}$?
What if $w \in \mathbb{R}$?
What if $w \in \mathbb{Z}_{\geq 0}$?

$(z \neq 0)$ $\quad z^0 = 1$, $\quad z^n = \underbrace{z \cdot z \cdots z}_{n\text{-times}}$

Def. $z^{1/m}$ as any complex no. s.t its m^{th} power is z.

Def $z^{m/n}$.

$\because$ $\mathbb{Q} \subseteq \mathbb{R}$ is dense, we can define z^{λ} $\forall \lambda \in \mathbb{R}$.

Then $z^w = e^{w \log z}$ select a branch of log.

Where does exp(z) series conv. ?

$\forall z \in \mathbb{C}$

Approach 1: Compute $\limsup \left|\frac{1}{n!}\right|^{1/n}$

Lem: In case $\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$ exists, then $R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$

Pf: Let us name $\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \alpha$

We will show $\alpha = R$

$\alpha \leq R$: If $\alpha = 0$, then $\alpha < R$.

Else, let $\alpha > 0$.

To show $\alpha \leq R$, show that $|z| < \alpha$, then $\sum_{n\geq 0} |a_n z|^n < \infty$

Choose $\lambda > 0$, s.t $|z| < \lambda < \alpha$

Find N s.t $\lambda < \left|\frac{a_n}{a_{n+1}}\right| \quad \forall n \geq N$

Def. $B = |a_N| r^N$

Consider $\quad |a_{N+1}| r^{N+1} = \underbrace{\frac{|a_{N+1}|}{|a_N|} r}_{\leq 1} |a_N| r^N \leq |a_N| r^N = B$

By ind^n, $\quad |a_n| r^n \leq B \quad \forall\ n \geq N+1$

$$\Rightarrow \quad |a_n z^n| = |a_n r^n| \left|\frac{z}{r}\right|^n \leq B \left|\frac{z}{r}\right|^n \quad \text{for } n \geq N.$$

$$\Rightarrow \quad \sum_{n=N}^{\infty} |a_n z^n| \leq B \sum_{n=N}^{\infty} \left|\frac{z}{r}\right|^n < \infty$$

$\Rightarrow$ Series converges

$\Rightarrow$ $z \in B(0, \alpha)$ makes series converge.

$\Rightarrow$ $\alpha \leq R$

<u>$\alpha \geq R$</u>: For showing $R \leq \alpha$, we show $|z| > \alpha$, series diverges.

Let $|z| > \alpha$. Like earlier, choose $r > \alpha$ s.t $|z| > r > \alpha$

Then $\exists\ N$ s.t $\forall\ n \geq N$, $\quad \dfrac{|a_n|}{|a_{n+1}|} \leq r$.

Choose $B = |a_N r^N|$

Like earlier, show that $|a_n \varrho^n| \geqslant B \quad \forall\, n \geqslant N$.

$\Rightarrow |a_n z^n| \geqslant B \left|\frac{z}{\varrho}\right|^n \to \infty$ as $n \to \infty$

$\Rightarrow$ Series diverges

$\Rightarrow \alpha \geqslant R$

Hence, $\alpha = R$.

For the exponential series,

$$R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \left|\frac{n+1}{z}\right| = \infty$$

Hence, the series conv. on the entire plane.

Recall from real analysis,

$$\exp(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!}$$

$$e = \exp(1) = \sum_{n=0}^{\infty} 1/n!$$

$$e^t = \left(\sum_{n=0}^{\infty} 1/n!\right)^t = \sum_{n=0}^{\infty} t^n/n!$$

<u>Idea</u>: 1. Taylor expansion is unique

2. Find $f^{(n)}(0)$ for $f(t) = e^t$

Define $\cos(z) = \frac{e^{iz} + e^{-iz}}{2}$, $\sin(z) = \frac{e^{iz} - e^{iz}}{2i}$

Power series expansion of $\cos(z)$.

$$e^{iz} = \sum_{n=0}^{\infty} \frac{(iz)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n-1}}{(2n-1)!}$$

So, $\cos(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \dots$

Sim, $\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots$

Differentiation in $\mathbb{C}$

Let $f : \Omega \to \mathbb{C}$ be a fn defined on some non-empty Ω st $\text{int}\, \Omega \neq \emptyset$

We define the derivative of f at $z \in \text{int}\, \Omega$

$$f'(z) = \lim_{\substack{h \to 0 \\ h \in \mathbb{C}}} \frac{f(z+h) - f(z)}{h}$$

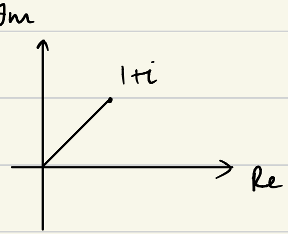
given that the limit exists.

Lem: Diff. $\Rightarrow$ Continuity

Pf.
$$\lim_{h\to 0} f(z+h)-f(z) = \lim_{h\to 0} \frac{f(z+h)-f(z)}{h}\cdot h$$
$$= \left(\lim_{h\to 0}\frac{f(z+h)-f(z)}{h}\right)\left(\lim_{h\to 0} h\right)$$
$$= f'(z)\cdot 0 = 0$$

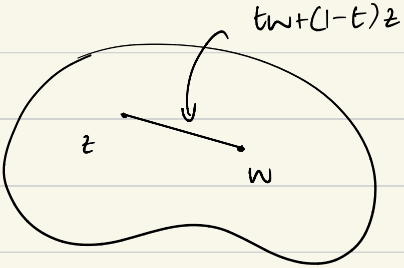
Path: Path inside X is the image of $[0,1]$ under a continuous map. i.e
$\gamma:[0,1]\to X$ be a cont. map.
$\gamma(t)$ is called a path from $\gamma(0)$ to $\gamma(1)$.

eg: $\gamma(t) = t+it$



Connected + open set = Path connected

<u>Lem</u>: If $\Omega \subseteq \mathbb{C}$ is a connected open set with $f: \Omega \to \mathbb{C}$ s.t $f'(z) = 0 \ \forall z \in \Omega$, then f is constant on Ω.



<u>Pf</u>: Suppose Ω is convex.

Consider $z \in \Omega$ & $w \in \Omega$

Let $z_0 = f(z)$

To prove : $f(w) = z_0$

Def. $g(t) = f(tw + (1-t)z)$

$$\frac{d}{dt} g(t) = \underbrace{f'(tw + (1-t)z)}_{0} \cdot (w - z) \qquad \text{(Chain Rule)}$$

$\Rightarrow$ $g : \mathbb{R} \to \mathbb{C}$ has $g'(t) = 0$ $\Rightarrow$ $g(0) = g(1)$

$\Rightarrow$ $f(w) = f(z)$

Recall, if $g : [a,b] \to \mathbb{R}$ is s.t $g'(x) = 0 \ \forall x \in (a,b)$ & g is cont. on $[a,b]$, then g is const.

General case: (Ω need not be convex)

Consider a path $\gamma(t)$ from z to w i.e $\gamma(0) = z$ & $\gamma(1) = w$.

$$g(t) = f(\gamma(t))$$

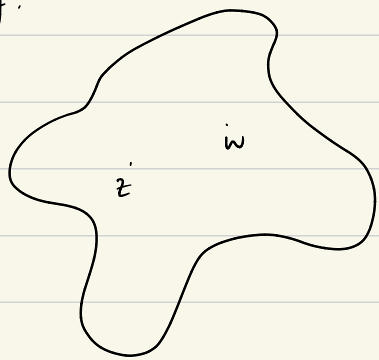
$$\Rightarrow \frac{d}{dt} g(t) = f'(\gamma(t)) \, \gamma'(t)$$

may not exist

Instead, we will try the following.

Let $z_0 = f(z)$

Define $A = \{ u \in \Omega : f(u) = z_0 \}$



$A \neq \emptyset$ since $z \in A$.

Claim: $A = \Omega$

By showing A is both open & closed.

<u>A is closed</u> : Let $\{u_n\}_{n\geq 1}$ be a Cauchy seq. in A.

i.e $f(u_n) = z_0 \ \forall \ n \geq 1$

$$\lim_{n\to\infty} f(u_n) = \lim_{n\to\infty} z_0 = z_0$$

$$=$$

$$f(\lim_{n\to\infty} u_n) = z_0 \qquad (\because f \text{ is diff, it is cont.})$$

$$\Rightarrow \quad \lim_{n\to\infty} u_n \in A$$

Hence, A is closed.

<u>A is open</u> : Consider $\epsilon > 0$, s.t $B_z(\epsilon) \subseteq \Omega$

<u>Claim</u>: $B_z(\epsilon) \subseteq A$

Consider any $a \in B_z(\epsilon)$.

$\because B_z(\epsilon) \subseteq \Omega$ is a convex set, the line

$\gamma(t) = at + (1-t)z \subseteq B_z(\epsilon)$

Then $g(t) = f(\gamma(t))$ satisfies

$$\frac{d}{dt} g(t) = \underbrace{f'(\gamma(t))}_{0} (a-z) = 0 \quad \Rightarrow \quad g \text{ is const.}$$

$$\Rightarrow \quad g(0) = g(1)$$

$$\Rightarrow \quad f(z) = f(a)$$

$\Rightarrow B_z(\epsilon) \subseteq A$

Hence, A is open.

Since $A \neq \emptyset$, therefore $A = \Omega$.

Differentiating power series

Consider the power series $f(z) = \sum_{n \geq 0} a_n (z-c)^n$ with $R > 0$

Claim: $f'(z) = \sum_{n \geq 1} a_n n (z-c)^{n-1}$

Pf: Let $g(z) = \sum_{n \geq 1} a_n n (z-c)^{n-1}$

We will show that radius of conv. of $g(z)$ is also R.

$$g(z) = \sum_{n \geq 0} a_{n+1} (n+1) (z-c)^n$$

Consider $\limsup_{n \to \infty} (n+1)^{1/n} = \limsup_{n \to \infty} e^{\log(n+1)^{1/n}}$

$$= e^{\limsup_{n \to \infty} \frac{\log(n+1)}{n}} \quad (\because e^x \text{ is increasing } \limsup f(x_n) = f(\limsup x_n))$$

$$= e^0 = 1$$

Consider $\limsup_{n\to\infty} |a_{n+1}|^{1/n}$ & the series

$$h(z) = \sum_{n\geq 0} a_{n+1} (z-c)^n$$

$\Rightarrow$ $f(z) = a_0 + h(z)(z-c)$

$\Rightarrow$ f converges $\Leftrightarrow$ h converges

$\Rightarrow$ RoC of $h = R$

So, $\limsup_{n\to\infty} |a_{n+1}|^{1/n} = 1/R$

Hence,

$$\limsup_{n\to\infty} |a_{n+1}(n+1)|^{1/n} = \limsup_{n\to\infty} \left(|a_{n+1}|^{1/n} \cdot |n+1|^{1/n}\right)$$

$$= \limsup_{n\to\infty} |a_{n+1}|^{1/n} \cdot \limsup_{n\to\infty} |n+1|^{1/n}$$

$$= 1/R$$

Let $z, w \in B_c(R)$ & consider $\dfrac{f(z)-f(w)}{z-w} - g(w)$

Then we take $\lim\limits_{z \to w}$ of above & show that it equals 0.

$$f(z) = \sum_{n=0}^{\infty} a_n(z-c)^n = \underbrace{\sum_{n=0}^{m} a_n(z-c)^n}_{S_m(z)} + \underbrace{\sum_{n=m+1}^{\infty} a_n(z-c)^n}_{R_m(z)}$$

$$\frac{f(z)-f(w)}{z-w} - g(w) = \frac{(S_m(z)+R_m(z)) - (S_m(w)+R_m(w))}{z-w} - g(w)$$

$$= \left(\frac{S_m(z)-S_m(w)}{z-w} - S_m'(w)\right) \to (1)$$

$$+ \underbrace{(S_m'(w) - g(w))}_{(2)} + \underbrace{\left(\frac{R_m(z)-R_m(w)}{z-w}\right)}_{(3)}$$

$$R_m(z) - R_m(w) = \sum_{n=m+1}^{\infty} a_n(z-c)^n - \sum_{n=m+1}^{\infty} a_n(w-c)^n$$

$$= \sum_{n=m+1}^{\infty} a_n\left[(z-c)^n - (w-c)^n\right]$$

$z_1 = z - c, \quad w_1 = w - c$

$$z_1^n - w_1^n = \underbrace{(z_1 - w_1)}_{(z-w)}\left(z_1^{n-1} + z_1^{n-2} w_1 + \cdots + z_1 w_1^{n-2} + w_1^{n-1}\right)$$

$$\left|\frac{R_m(z)-R_m(w)}{z-w}\right| \leqslant \sum_{n=m+1}^{\infty} |a_n| \left(|z_1|^{n-1}+|z_1|^{n-2}|w_1|+\ldots+|z_1||w_1|^{n-2}+|w_1|^{n-1}\right)$$

Choose $\delta>0$ s.t $|z-w|<\delta$ & $|z_1|, |w_1| \leqslant r < R$

$$\Rightarrow \left|\frac{R_m(z)-R_m(w)}{z-w}\right| \leqslant \underbrace{\sum_{n=m+1}^{\infty} |a_n|\, n r^{n-1}}_{\text{Tail of absolutely conv. series}}$$

$\leqslant \epsilon/3$ for $m>N_1$ for some $N_1 \in \mathbb{Z}_{>0}$

$$S_m(w) = \sum_{n=0}^{m} a_n (w-c)^n$$

$$\Rightarrow S_m'(w) = \sum_{n=1}^{m} a_n n (w-c)^{n-1}$$

$$S_m'(w) - g(w) = -\sum_{n=m+1}^{\infty} a_n \cdot n (w-c)^{n-1}$$

$\Rightarrow |S_m'(w)-g(w)| \leq \epsilon/3$ for $m \geqslant N_2$ for some $N_2 \in \mathbb{Z}_{>0}$

Choose $N = \max\{N_1, N_2\}$ s.t $\forall\, m \geq N$, (2) & (3) $< \epsilon/3$

Pick $m = N$, choose $\delta > 0$ s.t

$$\left| \frac{S_m(z) - S_m(w)}{z-w} - S_m'(w) \right| < \epsilon/3$$

when $|z-w| < \delta$

$g(z) = f'(z)$

$$g_2(z) = \sum_{n \geq 2} n(n-1)\, a_n (z-c)^{n-2}$$

$g_2(z)$ by same logic has ROC equals ROC of $g(z)$ $= R$ & that $g_2(z) = g'(z)$

By ind^n, $f(z)$ is smooth inside $B_c(R)$ with termwise differentiation.

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \quad \text{conv. everywhere.}$$

Also now, $\frac{d}{dz} \exp(z) = \sum_{n=1}^{\infty} \frac{n \cdot z^{n-1}}{n!} = \exp(z)$

By termwise diff., $\frac{d}{dz} \sin(z) = \cos(z)$

$\frac{d}{dz} \cos(z) = -\sin(z)$

For real t, $e^t = \exp(t)$ & also $\frac{d}{dt} e^t = e^t$

$\therefore$ Notationally, it makes sense to write $\exp(z) = e^z$

Ppts: $e^{a+b} = e^a \cdot e^b \quad \forall\, a, b \in \mathbb{C}$

Let $g(z) = e^{a-z} \cdot e^z$

$\Rightarrow g'(z) = -e^{a-z} \cdot e^z + e^{a-z} \cdot e^z = 0$

$g(z) = \text{const.} = g(0) = e^a \cdot e^0 = e^a \Rightarrow e^{a-z} \cdot e^z = e^a$

Take $A = a-z$, $B = z$ & $A+B = a \Rightarrow e^A \cdot e^B = e^{A+B}$

Show $\exp(\bar{z}) = \overline{\exp(z)}$.

For $\theta \in \mathbb{R}$, we get

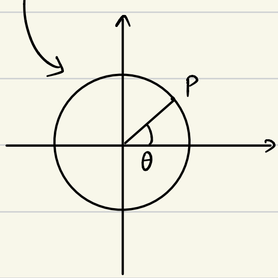
$$\begin{aligned} |\exp(i\theta)|^2 &= \exp(i\theta)\,\overline{\exp(i\theta)} \\ &= \exp(i\theta)\exp(\overline{i\theta}) \\ &= \exp(i\theta)\exp(-i\theta) \\ &= \exp(i\theta - i\theta) = 1 \end{aligned}$$

Also, $\exp(i\theta) = \cos\theta + i\sin\theta$

$\Rightarrow |\exp(i\theta)|^2 = \cos^2\theta + \sin^2\theta = 1$

So, $(\cos\theta, \sin\theta)$ satisfies the eq^n $x^2 + y^2 = 1$

Now, we need to justify the point P on the circle is given by $(\cos\theta, \sin\theta)$



θ = arc length of unit circle

$\because$ Given two fn^{n}s f & g s.t $f^2(t) + g^2(t) = 1$ does <u>not</u> imply that $(f(t), g(t))$ parametrize the circle. eg – $f(t) = t$, $g(t) = \sqrt{1-t^2}$

Def. $\gamma(t) = (\cos(t), \sin(t))$

$\cos^2(t) + \sin^2(t) = 1$

So, γ lies on the circle.

$$\text{Arc length} = \int_0^{\theta} |\gamma'(t)|\,dt = \int_0^{\theta} 1\,dt = \theta$$

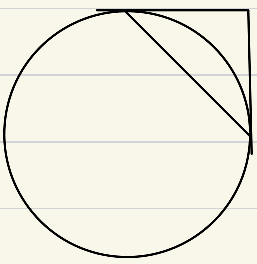
$$\text{Speed of motion} = |\gamma'(t)| = \sqrt{(-\sin(t))^2 + (\cos(t))^2} = 1$$

For $\gamma(t)$, even $|\gamma''(t)| < \infty$

This means we're only moving in one direction -
either clockwise or anticlockwise

$\gamma'(0) = (0,1) \Rightarrow$ we're moving anti-clockwise.

We just need to show that perimeter is finite.
So, we need to bound arc-length
by two convex curves.



This boundedness argument requires
circle to be a convex curve.

Def. as 2π, the smallest period of $\cos(t)$ & $\sin(t)$.

$$e^z = e^a \cdot e^{ib} = e^a(\cos(b) + i\sin(b)) \qquad \text{for } z = a+ib$$

$e^t : \mathbb{R} \to \mathbb{R}_{>0}$ is surjective

& $\cos(t) : \mathbb{R} \to [-1,1]$ is also surjective

Thus we may write, $z = re^{i\theta}$ for $r > 0$,

where θ is called the principal argument of z.

Not^n: $\arg(z) = \{ \theta + 2n\pi : n \in \mathbb{Z} \text{ \& } \theta \in \mathbb{R} \text{ s.t } \frac{z}{|z|} = e^{i\theta} \}$

Define for $z \neq 0$

Define $\operatorname{Arg}(z) = \theta$ for which $-\pi \leq \theta \leq \pi$ & $\theta \in \arg(z)$

log is defined as inverse fn^n of exp.

Domain of log is $\mathbb{C} \setminus \{0\}$

$$\log : \mathbb{C} \setminus \{0\} \to \mathbb{C}$$

$$\log(z) = \log(|z| e^{i\theta})$$

$$= \log(|z|) + i\theta$$

$$\log(z) = \{ \log(|z|) + i\theta \; : \; \theta \in \arg(z) \}$$
$$= \{ \log(|z|) + i(\theta + 2n\pi) : \; \theta = \operatorname{Arg}(z), \; n \in \mathbb{Z} \}$$

Def. $\operatorname{Log}(z) = \log(|z|) + i\theta$, where $\theta = \operatorname{Arg}(z)$

Cauchy - Riemann eq^{n}s

Let $f : \Omega \to \mathbb{C}$ be complex diff. i.e $\lim\limits_{h \to 0} \dfrac{f(z+h) - f(z)}{h}$ exists $\forall \; z \in \operatorname{int}(\Omega)$.

Let $f(x+iy) = u(x,y) + iv(x,y)$ where $u, v : \mathbb{R}^2 \to \mathbb{R}$

If $h \in \mathbb{R}$, $\lim\limits_{\substack{h \to 0 \\ h \in \mathbb{R}}} \dfrac{f(z+h) - f(z)}{h}$

$$= \lim_{\substack{h \to 0 \\ h \in \mathbb{R}}} \frac{[u(x+h,y) + iv(x+h,y)] - [u(x,y) + iv(x,y)]}{h}$$

$$= \lim_{h \to 0} \frac{u(x+h,y) - u(x,y)}{h} + i \lim_{h \to 0} \frac{v(x+h,y) - v(x,y)}{h}$$

$$= \frac{\partial u}{\partial x}(x,y) + i \frac{\partial v}{\partial x}(x,y)$$

$$= u_x(x,y) + i v_x(x,y)$$

If $h \in i\mathbb{R}$, replace h with ih, $h \in \mathbb{R}$.

$$f'(z) = \lim_{\substack{h \to 0 \\ h \in \mathbb{R}}} \frac{f(z+ih) - f(z)}{ih}$$

$$= \lim_{\substack{h \to 0 \\ h \in \mathbb{R}}} \frac{[u(x,y+h) + iv(x,y+h)] - [u(x,y) + iv(x,y)]}{ih}$$

$$= -i \lim_{h \to 0} \frac{u(x,y+h) - u(x,y)}{h} + \lim_{h \to 0} \frac{v(x,y+h) - v(x,y)}{h}$$

$$= -i\, u_y(x,y) + v_y(x,y)$$

Comparing $\mathbb{R}$ & $i\mathbb{R}$ parts,

$$\Rightarrow \boxed{\begin{aligned} u_x(x,y) &= v_y(x,y) \\ u_y(x,y) &= -v_x(x,y) \end{aligned}}$$

known as the Cauchy-Riemann eq^{n}s.

If $u, v : \mathbb{R}^2 \to \mathbb{R}$ are s.t u, v satisfy CR eq^{n}s & $u, v \in C^2(\mathbb{R}^2)$
then $f(x+iy) = u(x,y) + iv(x,y)$ is $\mathbb{C}$-diff.

(Hint: If $u \in C^2(\mathbb{R}^2)$, then $u_{xy} = u_{yx}$)

Let $f : \Omega \to \mathbb{C}$ s.t u or v is a const., where
$f : \Omega \to \mathbb{C}$ is diff. & Ω is open & connected.

Then, $f'(z) = u_x + iv_x$
Let $u(x,y) = \text{const.} \Rightarrow u_x = u_y = 0$
$\Rightarrow f'(z) = 0 + iv_x = 0 - iu_y = 0$
$\Rightarrow f$ is a const. on Ω.

$\Rightarrow f : \Omega \to \mathbb{R} \subseteq \mathbb{C}$ is diff. $\Rightarrow f = \text{const.}$

If $f = u + iv$ & f is $\mathbb{C}$-diff., then $u(x,y)$ & $v(x,y)$ are harmonic fn's i.e they satisfy the diff. eqn,

$$\left(\frac{\partial^2}{\partial^2 x} + \frac{\partial^2}{\partial^2 y}\right) u = 0$$

<u>Assumption 1:</u>

To show this, we assume that we know if f is $\mathbb{C}$-diff., then f' is also $\mathbb{C}$-diff.

$\Rightarrow \quad u, v \in C^2(\mathbb{R}^2)$

Notice,
$$\begin{aligned} u_{xx} + u_{yy} &= (u_x)_x + (u_y)_y \\ &= (v_y)_x + (-v_x)_y \\ &= v_{yx} - v_{xy} \qquad (\because \text{ If } v \in C^2(\mathbb{R}^2),\ v_{xy} = v_{yx}) \\ &= 0 \end{aligned}$$

Sim. for v, $\quad v_{xx} + v_{yy} = 0$

Q1. Given $u \in C(\mathbb{R}^2)$ & $u_{xx} + u_{yy} = 0$, does there exist $v \in C(\mathbb{R}^2)$ s.t $u + iv$ is $\mathbb{C}$-diff.

Q2. Given $u \in C(\mathbb{R}^2)$ & $u_{xx} + u_{yy} = 0$, then $u \in C^\infty(\mathbb{R}^2)$

Assumption 2: If $\Omega \in \mathbb{C}$ is open & connected s.t $f : \Omega \to \mathbb{C}$ is diff. on Ω, then $\exists g : \Omega \to \mathbb{C}$ s.t $f = g'$.

A1. Given $u \in C^2(\mathbb{R}^2)$, def. $h : \Omega \to \mathbb{C}$ by
$$h(x+iy) = u_x(x,y) - iu_y(x,y)$$

CR eq^{n}s: 1. $\because$ $u_{xx} + u_{yy} = 0$
$$\Rightarrow (u_x)_x = u_{xx} = -u_{yy} = (-u_y)_y$$

2. $\because$ $u \in C^2(\mathbb{R}^2)$
$$\Rightarrow u_{xy} = u_{yx} \Rightarrow (u_x)_y = -(-u_y)_x$$

Hence, u_x & $-u_y$ satisfy CR eq^{n}s.

Indeed, $h : \Omega \to \mathbb{C}$ is $\mathbb{C}$-diff.

By Assumption 2, $\exists f : \Omega \to \mathbb{C}$ s.t $f' = h$

Let $f = u_1 + iv_1$

Fix pt. $z_0 \in \Omega$ & fudge u_1 s.t $u_1(z_0) = u(z_0)$

Then, $f' = (u_1)_x + i(v_1)_x$ $\quad -(1)$

$= (v_1)_y - i(u_1)_y$ (by CR eqns) $\quad -(2)$

Also, $f' = h = u_x - iu_y$ $\quad -(3)$

(1) & (3) $\Rightarrow$ $(u_1)_x = u_x \Rightarrow (u_1 - u)_x = 0$

(2) & (3) $\Rightarrow$ $(u_1)_y = u_y \Rightarrow (u_1 - u)_y = 0$

$\Rightarrow u_1 - u = \text{const.}$

$\Rightarrow u_1(z_0) - u(z_0) = \text{const.}$

$\Rightarrow \text{const.} = 0$

Hence, f is diff. & $\text{Re}(f) = u$.

Path integral

Let $\gamma : [a,b] \to \mathbb{C}$ be a cont. map be a path.

We call γ to be smooth if $\gamma'(t)$ exists $\forall\ t \in (a,b)$

& also $\lim\limits_{\substack{h\to 0 \\ h>0}} \dfrac{\gamma(a+h)-\gamma(a)}{h}$ & $\lim\limits_{\substack{h\to 0 \\ h<0}} \dfrac{\gamma(b+h)-\gamma(b)}{h}$ exist,

and $\gamma'(t) \neq 0$ for $t \in [a,b]$ & $\gamma \in C^1([a,b])$

eg: $\gamma(t) = (\cos(t), \sin(t))$

Two parametrizations $\gamma_1 : [a,b] \to \mathbb{C}$ & $\gamma_2 : [c,d] \to \mathbb{C}$ are called equiv. iff $\exists$ a bij. $\sigma : [a,b] \to [c,d]$ with σ monotonically increasing with $\sigma \in C^1([a,b])$ &
$\gamma_2(\sigma(t)) = \gamma_1(t)\quad \forall\ t \in [a,b]$

Family of equivalent parametrization defines a smooth curve $\gamma \subseteq \mathbb{C}$

Note: $\gamma_1(t) = (\cos(t), \sin(t))$, $t \in [0, 2\pi]$

$\gamma_2(t) = (\cos(t), \sin(t))$, $t \in [0, 4\pi]$

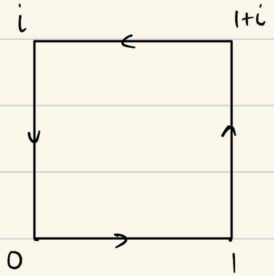
$\gamma_1 \not\sim \gamma_2$

Closed curve : $\gamma_1(a) = \gamma_1(b)$

Simple curve : If $\gamma_1(t) = \gamma_2(t)$ iff $t_1 = t_2$ or
if $t_1 \neq t_2$, then $t_1 = a$ & $t_2 = b$

Piecewise smooth curve : If $\exists\, n \in \mathbb{Z}_{>0}$ s.t for
$\gamma : [a, b] \to \mathbb{C}$, $\exists$ a partition $a = a_0 < a_1 < \dots < a_n = b$ s.t
γ is smooth on each of $[a_i, a_{i+1}]$ $\forall\ 0 \leq i \leq n-1$

eg:



Def. $$\int_\gamma f(z)\, dz = \int_a^b f(\gamma(t)) \cdot \gamma'(t)\, dt$$

Note: This integral should be indep. of the parametrization.
We should have $\gamma_1 \sim \gamma_2 \Rightarrow \int_{\gamma_1} f(z)\, dz = \int_{\gamma_2} f(z)\, dz$

Check: $$\int_a^b f(\gamma_1(t)) \cdot \gamma_1'(t)\, dt = \int_c^d f(\gamma_2(t)) \cdot \gamma_2'(t)\, dt$$

$\gamma_1 \sim \gamma_2$ means $\exists\, \sigma : [a,b] \to [c,d]$ s.t $\gamma_2(\sigma(t)) = \gamma_1(t)$

$$\int_a^b f(\gamma_1(t)) \cdot \gamma_1'(t)\, dt = \int_a^b f(\gamma_2(\sigma(t))) \cdot \gamma_2'(\sigma(t))\, \sigma'(t)\, dt$$

$$= \int_c^d f(\gamma_2(s)) \cdot \gamma_2'(s)\, ds$$

Let $s = \sigma(t)$

$ds = \sigma'(t)\, dt$

$\int_a^b \to \int_c^d$

Length of curve $:= \int_a^b |\gamma'(t)|\, dt$

Inequality: $\left| \int_\gamma f(z)\, dz \right| \leq \sup_{z \in \gamma} |f(z)| \cdot \text{length}(\gamma)$

$\underline{\text{Pp}^n}$: If $f : \Omega \to \mathbb{C}$ is s.t $\exists F : \Omega \to \mathbb{C}$ with $F'(z) = f(z)$ $\forall z \in \mathbb{C}$ then $\int_\gamma f(z)\, dz = F(w_1) - F(w_2)$ where $\gamma : [a,b] \to \mathbb{C}$ & $w_1 = \gamma(a)$
$w_2 = \gamma(b)$

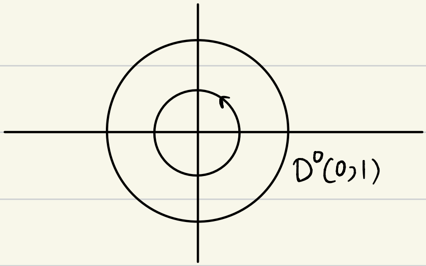
$\underline{\text{Pf}}$: $$\int_\gamma f(z)\, dz = \int_a^b f(\gamma(t)) \cdot \gamma'(t)\, dt = \int_a^b F'(\gamma(t)) \cdot \gamma'(t)\, dt$$
$$= \int_a^b \frac{d}{dt}(F(\gamma(t))\, dt = F(\gamma(b)) - F(\gamma(a))$$

Thus, if f has an antiderivative (primitive) i.e $\exists F$ s.t $F' = f$ & γ is closed curve, then $\int_\gamma f = 0$

Note that this implies that $1/z$ does not have a primitive in $D^\circ(0,1) = \{ z \in \mathbb{C} : |z| < 1 \}$

Why? Show $\exists$ closed γ s.t $\int_\gamma f \neq 0$ for $f = 1/z$

$$\gamma(t) = \left(\frac{1}{2}\cos(t), \frac{1}{2}\sin(t) \right)$$
$$= \frac{1}{2} e^{it}, \quad t \in [0, 2\pi]$$



$$\gamma'(t) = \frac{i}{2} e^{it}$$

$$\int_\gamma f\,dz = \int_\gamma \frac{1}{z}\,dz = \int_0^{2\pi} \frac{1}{\frac{1}{2}e^{it}} \cdot \frac{ie^{it}}{2}\,dt = 2\pi i$$

This happens because $\frac{d}{dz}\,\mathrm{Log}(z) = \frac{1}{z}$ & $\mathrm{Log}(z)$ has a branch cut on -ve x-axis.

From this, we also have $f' = 0$ & $f = \text{const.}$ on Ω (open & connected)

Let $z_0, w \in \Omega$ s.t $\exists$ a piecewise smooth path γ from z_0 to w, then

$$f(w) - f(z_0) = \int_\gamma f'(z)\,dz = \int_\gamma 0\,dz = 0$$

$$\Rightarrow \quad f(w) = f(z_0)$$

Pf: $A = \{ w \in \Omega : \exists$ piecewise smooth γ from z_0 to $w \}$

$A^c = \Omega \setminus A$

A is open: Let $\epsilon > 0$ s.t $B_{z_0}(\epsilon) \subseteq \Omega$.

Then $B_{z_0}(\epsilon) \subseteq A$

($\because B_{z_0}(\epsilon)$ is convex, $\exists$ straight line path from z_0 to w which is clearly smooth)

Consider $\epsilon > 0$ s.t for $w \in A$, $B_w(\epsilon) \subseteq \Omega$.

Then $\exists$ smooth path from w to $w_1 \in B_w(\epsilon)$ & then the concatenated path $\tilde{\gamma} \circ \gamma$ connects z_0 to w_1.

$\Rightarrow w_1 \in A \Rightarrow B_w(\epsilon) \subseteq A$

A^c is open: Consider $w \in A^c$ & choose $\epsilon > 0$ s.t $B_w(\epsilon) \subseteq \Omega$

If $w_1 \in B_w(\epsilon)$ s.t $w \in A$, then $\exists$ smooth path $\tilde{\gamma}$ from w_1 to $w \Rightarrow w$ can be connected to $z \Rightarrow w \in A \rightarrow$ Contd^n.

So, A & A^c are both open & $A \cup A^c = \Omega$.

However, Ω is connected $\Rightarrow A = \emptyset$ or $A^c = \emptyset$

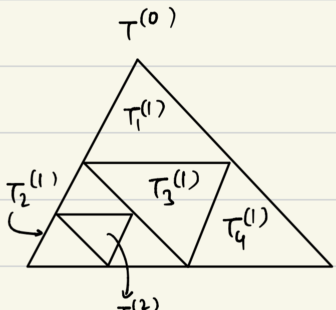
$\because z_0 \in A \Rightarrow A \neq \emptyset \Rightarrow A^c = \emptyset \Rightarrow A = \Omega$.

Thm: (Goursat's Thm)

If $\Omega \subseteq \mathbb{C}$ & T is a triangle (closed set, which is ∂T with interior included) & f is diff. on Ω, then

$$\int_{\partial T} f(z)\,dz = 0$$

Pf: Here, $T^{(0)} \sim T_1^{(1)} \sim T_2^{(1)} \sim T_3^{(1)} \sim T_4^{(1)}$
(similar)



$$\int_{\partial T^{(0)}} f(z)dz = \int_{\partial T^{(0)}} f = \int_{\partial T_1^{(1)}} f + \int_{\partial T_2^{(1)}} f + \int_{\partial T_3^{(1)}} f + \int_{\partial T_4^{(1)}} f$$

$\therefore$
$$\left| \int_{\partial T^{(0)}} f \right| \leq \sum_{i=1}^{4} \left| \int_{T^{(1)}} f \right| \leq 4 \left| \int_{T_i^{(1)}} f \right|$$
$$\leq 4 \times 4 \left| \int_{T^{(2)}} f \right| \leq 4^n \left| \int_{T^{(n)}} f \right|$$

$\therefore$ Perimeter of $\partial T^{(n)} = \dfrac{P}{2^n}$

Diameter of $\partial T^{(n)} = \dfrac{d}{2^n}$

Note, $\quad T^{(0)} \supseteq T^{(1)} \supseteq \dots \Rightarrow \lim_{n\to\infty} \operatorname{diam}(T^{(n)}) = 0$

$$\Rightarrow \bigcap_{n=0}^{\infty} T^{(n)} = \text{singleton set}$$
$$= \{z_0\} \subseteq \Omega$$

$\because$ f is diff. at z_0.

$\Rightarrow f(z) = f(z_0) + f'(z_0)(z-z_0) + \varphi(z)(z-z_0)$

where $z \in B(z_0, \epsilon) \subseteq \Omega$ & $\lim_{z\to z_0} \varphi(z) = 0$

Now,
$$\left|\int_{\partial T^{(n)}} f(z)\,dz\right| = \left|\underbrace{\int_{\partial T^{(n)}} f(z_0)\,dz}_{0} + \underbrace{\int_{\partial T^{(n)}} f'(z_0)(z-z_0)\,dz}_{\left.\frac{(z-z_0)^2}{2}\right|_{\partial T^{(n)}}} + \int_{\partial T^{(n)}} \varphi(z)(z-z_0)\,dz\right|$$

b/c primitive exists & b'ry is closed

$\int 1\cdot dz = z$

$= 0$

$$= \left|\int_{\partial T^{(n)}} \varphi(z)(z-z_0)\,dz\right|$$

$$\leq \sup_{z\in\partial T^{(n)}} |\varphi(z)(z-z_0)| \cdot \text{length of } \partial T^{(n)}$$

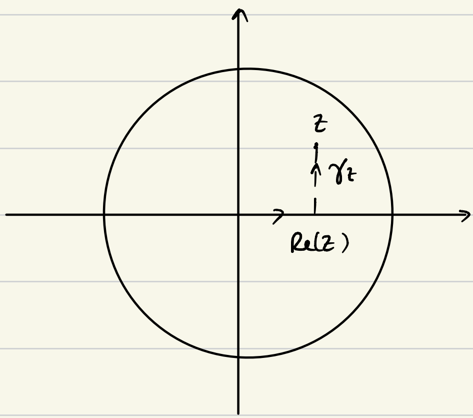
$$\leq \sup_{z\in\partial T^{(n)}} |\varphi(z)| \cdot \frac{d}{2^n} \cdot \frac{p}{2^n}$$

$\because \sup_{z \in T^{(n)}} |\varphi(z)| = \epsilon_n \to 0$ as $n \to \infty$

$$\therefore \left| \int_{\partial T^{(0)}} f \right| \leq 4^n \frac{\epsilon_n d \cdot p}{4^n} = \epsilon_n dp \to 0 \quad \text{as } n \to \infty.$$

$$\Rightarrow \left| \int_{\partial T^{(0)}} f \right| = 0 \Rightarrow \int_{\partial T^{(0)}} f = 0$$

<u>Thm</u>: A diff. fx^n in an open disc (convex set) has a primitive i.e if $\Omega = D^\circ(c, R)$ & f is diff on Ω then $\exists\, F : \Omega \to \mathbb{C}$ s.t $F'(z) = f(z)\ \forall\, z \in \Omega$.



<u>Pf</u>: Def. $F(z) = \int_{\gamma_z} f(w)\, dw$

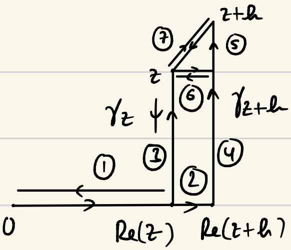
<u>Goal</u>: 1. F is diff.

2. $F'(z) = f(z)$ i.e

$$\lim_{h \to 0} \frac{F(z+h) - F(z)}{h} = f(z)$$

$\therefore \quad F(z+h) - F(z)$

$$= \underbrace{\int_{\substack{\text{Rect.}\\ 2,3,4,6}} f}_{0} + \underbrace{\int_{\substack{\Delta \text{ made}\\ \text{of } 5,6,7}} f}_{0} + \int_{\eta} f$$



$$\Rightarrow F(z+h) - F(z) = \int_{\eta} f(w)\, dw$$

f is diff. at $z \Rightarrow f$ is cont. at z.

$$\Rightarrow \quad f(w) = f(z) + \psi(w) \quad \text{where } \lim_{w\to z} \psi(w) = 0$$

$$\Rightarrow F(z+h) - F(z) = \int_{\eta} f(w)\, dw = \int_{\eta} f(z)\, dw + \int_{\eta} \psi(w)\, dw$$

$$= f(z)\, h + \int_{\eta} \psi(w)\, dw \quad \left(\because \int 1\, dw = h\right)$$

$$\Rightarrow \frac{F(z+h) - F(z)}{h} = f(z) + \frac{1}{h}\int_{\eta} \psi(w)\, dw$$

$$\because \quad \left| \frac{1}{h} \int_{\eta} \psi(w)\, dw \right| \leq \frac{1}{h} \sup_{w\in\eta} |\psi(w)|\, h$$

$$= \sup_{w\in[z,z+h]} |\psi(w)| \to 0 \quad \text{as } h \to 0$$

$$\therefore \quad \lim_{h\to 0} \frac{F(z+h) - F(z)}{h} = f(z) \quad \Rightarrow \quad F'(z) = f(z)$$

<u>Rem</u>: The same construction can be used for any convex set.

Result: Given an open connected $\Omega \subseteq \mathbb{C}$ & two pts. $z, w \in \mathbb{C}$, $\exists$ a polygon line (a concatenation of finitely many horizontal & vertical paths) that connects z & w.

Pf: Fix z.

Def. $A = \{w \in \Omega : \exists$ a polygon line connecting z & $w\}$

$A^c = \Omega \setminus A$

Goal: A is open

Choose $w \in A$. Then choose $\epsilon > 0$ s.t $B(w, \epsilon) \subseteq \Omega$ & show $B(w,\epsilon) \subseteq A$.

$\Rightarrow \forall\, w_1 \in B(w, \epsilon)$, $w_1 \in A \Rightarrow B(w,\epsilon) \subseteq A$

$\Rightarrow A$ is open (∵ $\exists$ polygon line connecting w_1 & w)

Goal: A^c is open

Pick $w \in A^c$. Choose $\epsilon > 0$ s.t $B(w,\epsilon) \subseteq \Omega$. Let $w_1 \in B(w,\epsilon)$

If $w_1 \in A$, then $w \in A$, which is a contdn.

$\Rightarrow B(w,\epsilon) \subseteq A^c \Rightarrow A^c$ is open

∵ $A \cup A^c = \Omega$ (connected) $\Rightarrow A = \emptyset$ or $A^c = \emptyset$

But since $w \in A$, $A \neq \emptyset \Rightarrow A = \Omega$.

Cauchy's Thm for a convex domain

$\int_\gamma f(z)\,dz = 0$ where $f : \Omega \to \mathbb{C}$ is diff.,

Ω is convex & open & γ is a closed curve.

More generally, we want to establish this for simply-connected domains

(& for some non-simply connected domains)

Simply-connected domain:

Let $\gamma_1 : [0,1] \to \Omega$, $\gamma_2 : [0,1] \to \Omega$ (where γ_1 & γ_2 are cont.)

s.t $\gamma_1(0) = \gamma_2(0)$ & $\gamma_1(1) = \gamma_2(1)$

We say γ_1 & γ_2 are homotopic, denoted by $\gamma_1 \sim \gamma_2$

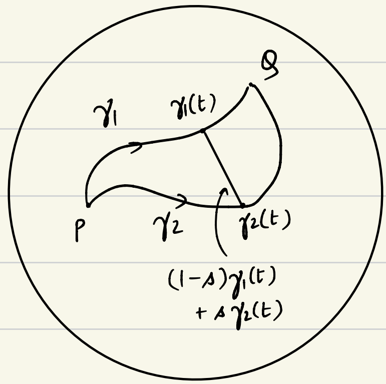
if there exists a cont. fx^n $F : \underbrace{[0,1]}_{\text{time}} \times \underbrace{[0,1]}_{\text{curve}} \to \Omega$ s.t

$$F(0,t) = \gamma_1(t)$$
$$F(1,t) = \gamma_2(t)$$

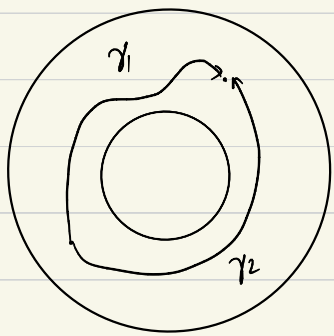
We say Ω is a simply connected domain if given 2 paths γ_1 & γ_2 with $\gamma_1(0) = \gamma_2(0)$ & $\gamma_1(1) = \gamma_2(1)$ then $\gamma_1 \sim \gamma_2$

eg: 1. Disc $(0,1)$

$$F(s,t) = (1-s)\gamma_1(t) + s\gamma_2(t)$$



2. non-example
Annulus



Lem: Let $D' = \mathring{D}(0,1) \setminus \{\zeta_1, \dots, \zeta_n\}$ where $\zeta_1, \dots, \zeta_n \in \mathbb{C}$.

Let diff. $f : D' \to \mathbb{C}$ s.t $\lim_{z \to \zeta_i} (z - \zeta_i) f(z) = 0 \quad \forall\ 1 \le i \le n$.

Then $\int_\gamma f = 0$ for any closed curve $\gamma \subseteq D'$

Rem: Points ζ_i with $\lim_{z \to \zeta_i} (z - \zeta_i) f(z) = 0$ are called Removable singularities of f.

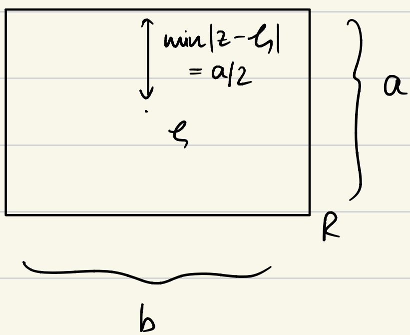
Lem: (Supporting)

If $R' = R \setminus \{\zeta_1, \dots, \zeta_n\}$ where R is rectangle in $\mathbb{C}$ &

$f : R' \to \mathbb{C}$ is diff. s.t $\lim_{z \to \zeta_i} (z - \zeta_i) f(z) = 0 \quad \forall\ 1 \le i \le n$,

then $\int_{\partial R} f = 0$.

Pf: WLOG, assume $n = 1$ i.e. only one exceptional pt.



where R is small i.e given $\epsilon > 0$ we have $|f(z)| \le \dfrac{\epsilon}{|z - \zeta|}$ for $z \in \partial R$

(can choose square)

$$\Rightarrow \left| \int_{\partial R} f(z)\,dz \right| \leq \sup_{z\in\partial R} |f(z)| \cdot \text{length}(\partial R)$$

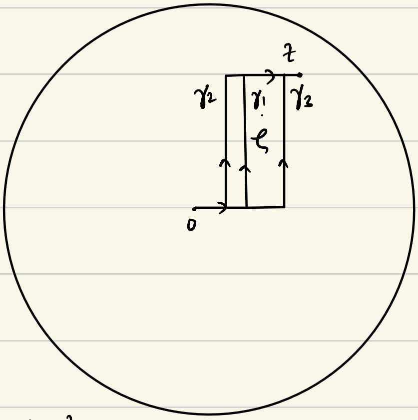
$$\leq \epsilon \sup_{z\in\partial R} \frac{1}{|z-\zeta_i|} \cdot \text{length}(\partial R)$$

$$\leq \epsilon \cdot \frac{2}{a} \cdot 4a = 8\epsilon$$

Since $\epsilon > 0$ is arbitrary, $\displaystyle\int_{\partial R} f = 0$

<u>Pf</u> : Possibility 1 - Center of disc is <u>not</u> any ζ_i

Def. $F(z) = \displaystyle\int_{\gamma} f(w)\,dw$



By supporting lem., F is well-defined

i.e $\displaystyle\int_{\gamma_1} f = \int_{\gamma_2} f = \int_{\gamma_3} f$

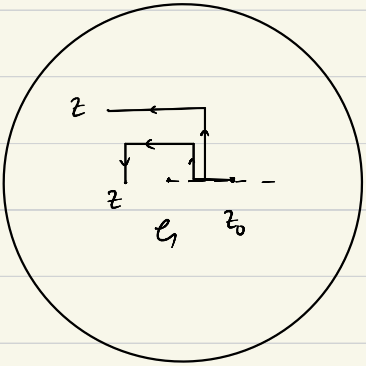
Choose $\epsilon > 0$ s.t $B(z,\epsilon) \subseteq D^{\circ}(0,1) \setminus \{\zeta_1, \dots, \zeta_n\}$

& choose h s.t $z+h \in B(z,\epsilon)$.

Then $F(z+h) - F(z) = \int_{\eta} f(w)\,dw = h \cdot f(z) + \psi_h(z) \cdot h$

Possibility 2: Center is an expectional pt.

One may generalize this construction to any simply connected domain.



Jordan-Curve Thm:

Given a simple closed curve $\gamma \subseteq \mathbb{C}$, $\mathbb{C} \setminus \{\gamma\}$ has exactly two connected components.

Winding number: Let $\gamma \subseteq \mathbb{C}$ be a closed piecewise smooth curve & $a \in \mathbb{C} \setminus \{\gamma\}$.

Then $\int_{\gamma} \frac{1}{z-a}\,dz = 2\pi i n$ for some integer n.

n is called the winding number.

Pf: Let $\gamma : [\alpha, \beta] \to \mathbb{C}$ with $\gamma(\alpha) = \gamma(\beta)$

Def. $h(t) = \displaystyle\int_\alpha^t \frac{\gamma'(s)}{\gamma(s) - a} \, ds$

(Here, assume γ is smooth)

Then by FTC, $h'(t) = \dfrac{\gamma'(t)}{\gamma(t) - a}$ for $t \in (\alpha, \beta)$

Consider $g(t) = e^{-h(t)}(\gamma(t) - a)$

Then $g'(t) = -h'(t) e^{-h(t)}(\gamma(t) - a) + e^{-h(t)}(\gamma'(t))$

$$= e^{-h(t)}(-\gamma'(t) + \gamma'(t)) = 0$$

Then $g(\alpha) = g(\beta)$.

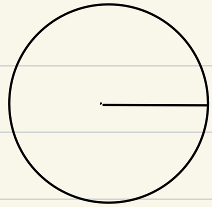
Also, $g(\alpha) = e^0 \cdot (\gamma(\alpha) - a)$

$g(\beta) = e^{-h(\beta)}(\underbrace{\gamma(\beta)}_{\gamma(\alpha)} - a)$

$$\Rightarrow e^{-h(\beta)} = \left(\frac{\gamma(\alpha) - a}{\gamma(\beta) - a} \right) = 1$$

$$\Rightarrow h(\beta) = 2\pi i n = \int_\alpha^\beta \frac{\gamma'(s) \, ds}{\gamma(s) - a} = \int_{\gamma_1} \frac{dz}{z - a}$$

eg: 1. $\gamma = \partial D(a, R)$



Parametrize γ as

$\gamma(t) = a + Re^{it}$, $t \in [0, 2\pi)$

Then $$\frac{1}{2\pi i}\int_0^{2\pi} \frac{1}{Re^{it}} Rie^{it}\,dt = 1$$

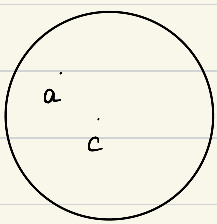
$z = a + Re^{it}$

$dz = Rie^{it}\,dt$

2. If $\gamma(t) = a + Re^{it}$ but $t \in [0, 4\pi)$, then

$$\frac{1}{2\pi i}\int_0^{2\pi} \frac{1}{z-a}\,dz = 2$$

3. $\gamma = \partial D(c, R)$, $a \neq c$



$$\frac{1}{2\pi i}\int_0^{2\pi} \frac{1}{c + Re^{it} - a} iRe^{it}\,dt = ?$$

<u>Ppⁿ</u>: The winding no. of every point in a given connected component of $\mathbb{C} \setminus \{\gamma\}$ is the same.

<u>Recall</u>: Given an open connected set Ω, any two pts. a & b in Ω can be connected by a path made of finitely many lines.

<u>Pf</u>: Let a, b be two pts. in the same connected comp. Ω_1 of $\mathbb{C} \setminus \{\gamma\}$ s.t the line segment $L(a,b)$ lies inside Ω_1.

<u>Goal</u>:
$$\frac{1}{2\pi i} \int_\gamma \frac{dz}{z-a} = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z-b}$$

i.e.
$$\int_\gamma \left(\frac{1}{z-a} - \frac{1}{z-b} \right) dz = 0.$$

<u>Strategy</u>: Show that $\frac{1}{z-a} - \frac{1}{z-b}$ has a primitive in Ω_1.

Consider the fxn $g(z) = \log\left(\frac{z-a}{z-b}\right)$

When $z \in \Omega \setminus L(a,b)$, then $\frac{z-a}{z-b} \in \mathbb{C}\setminus(-\infty, 0]$

line joining a & b

$\left(\because \text{ otherwise } \frac{z-a}{z-b} = -\lambda \Rightarrow z = \frac{1}{1+\lambda} a + \frac{\lambda}{1+\lambda} b \in L(a,b) \right)$

log is diff. on $\mathbb{C}\setminus(-\infty,0]$ & derivative of $\log(w)$ is $\frac{1}{w}$

$\Rightarrow \frac{1}{z-a} - \frac{1}{z-b}$ has a primitive, which we just

said was $g(z) = \log\left(\frac{z-a}{z-b}\right)$ and $g(z)$ is cts. & diff. on $\mathbb{C}\setminus(-\infty,0]$

$$\Rightarrow \int_\gamma \frac{1}{z-a} - \frac{1}{z-b} = 0 \quad \text{for } \gamma \text{ closed}$$

Cor: The winding no. of $a \notin D(c,R)$ about $\gamma = \partial D(c,R)$ equals 0.

Pf: Consider $n(\gamma,a) = \dfrac{1}{2\pi i} \displaystyle\int_{\partial D(c,R)} \dfrac{1}{z-a}\, dz$

$$= \frac{1}{2\pi i} \int_{\partial D(c,R)} \frac{1}{z-b}\, dz$$

where $b \in \mathbb{C}$ s.t $|b|$ is sufficiently large.

Then $n(\gamma,a) \leq \dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} \left| \dfrac{1}{c-b+Re^{it}} \right| \cdot |iRe^{it}|\, dt$

$$\leq \frac{1}{2\pi} \frac{2\pi R}{|b|/2} = \frac{2R}{b} < \epsilon \quad \text{for } |b| > \frac{2R}{\epsilon}$$

(Observe $|c-b+Re^{it}| \geq \big||b| - \underbrace{|c+Re^{it}|}_{\leq |c|+R}\big|$

Choose $b \geq 2(|c|+R)$)

Cauchy's Integral formula

(for a disc)

Let f be a diff. fn^n def. on the open disc $D^\circ(C,R)$.
Let $\{\gamma\} \subseteq D^\circ(C,R)$ & $a \in D^\circ(C,R) \setminus \{\gamma\}$

Then $$\frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-a}\, dz = f(a) \cdot n(\gamma, a)$$

Pf: Recall that if g is diff. on $D^\circ(C,R) \setminus \{e_1, \dots, e_n\}$ and $\lim_{z \to e_i} (z - e_i)\, g(z) = 0$, then $\int_\gamma g(z)\, dz = 0$

where $\{\gamma\} \subseteq D^\circ(C,R) \setminus \{e_1, \dots, e_n\}$

We apply this to $F(z) = \dfrac{f(z) - f(a)}{z-a}$

$$\Rightarrow \lim_{z \to a} (z-a) F(z) = \lim_{z \to a} f(z) - f(a) = 0$$

& $F(z)$ is diff. on $D^\circ(C,R) \setminus \{a\}$ because $f(z)$ is diff. there & $\dfrac{1}{z-a}$ is diff. there

$$\Rightarrow \int_\gamma f(z)\,dz = 0 \quad \Rightarrow \quad \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}\,dz = \frac{1}{2\pi i}\int_\gamma \frac{f(a)}{z-a}\,dz$$

$$= f(a)\cdot n(\gamma, a)$$

<u>Thm</u>: If $f(z)$ is diff. on Ω, then it is smooth on Ω.

<u>Approach</u>: We prove Cauchy's Integral formula

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_{\partial D} \frac{f(w)}{(w-z)^{n+1}}\,dw$$

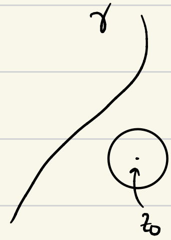
where ∂D is the b'ry of a disc $D \subseteq \Omega$ traversed counterclockwise once & $z \in D^{o}$.

<u>Note</u>: Need to justify diff. under the integral.

We prove a slightly more general result.
If γ is a closed path & φ is cts. on $\{\gamma\}$, then

$F_n = \int_\gamma \frac{\varphi(w)}{(w-z)^n}\,dw$ is diff. & $F_n'(z) = n F_{n+1}(z)$.

Pf : By ind^n.



Base case : $n=1$

Choose δ small enough s.t
$B(z,\delta) \subseteq \mathbb{C} \setminus \{\gamma\}$

Goal : $\displaystyle\lim_{z \to z_0} \frac{f_1(z) - f_1(z_0)}{z - z_0}$ exists.

Take $z \in B_{z_0}(\delta/2)$

Then $\text{dist}(\gamma, B_{z_0}(\delta/2)) \geqslant \delta/2$

To get a grasp, let's show $f_1(z)$ is cts. at z_0.

Consider $\displaystyle f_1(z) - f_1(z_0) = \int_\gamma \frac{\varphi(w)}{w-z}\,dw - \int_\gamma \frac{\varphi(w)}{w-z_0}\,dw$

$$= \int_\gamma \frac{\varphi(w)(z-z_0)}{(w-z)(w-z_0)}\,dw$$

$$= (z-z_0) \underbrace{\int_\gamma \frac{\varphi(w)}{(w-z)(w-z_0)}\,dw}_{\leq \text{length}(\gamma) \sup_{w \in \gamma} |\varphi(w)| \cdot \frac{2}{\delta^2}}$$

$\left(\begin{array}{l} \because\ |w - z_0| \geqslant \delta \\ |w - z| \geqslant \delta/2 \end{array} \right)$

By Dominated convergence thm,

$$\lim_{z \to z_0} \frac{F_1(z) - F_1(z_0)}{z - z_0} = \int_\gamma \lim_{z \to z_0} \frac{\varphi(w)}{(w-z)(w-z_0)} \, dw$$

$$= \int_\gamma \frac{\varphi(w)}{(w-z_0)^2} \, dw = F_2(w).$$

For ind^n, assume $F_{n-1}'(z) = (n-1) F_n(z)$

Goal: $F_n'(z) = n F_{n+1}(z)$

T'fore consider,

$$F_n(z) - F_n(z_0) = \int_\gamma \frac{\varphi(w) \, dw}{(w-z)^n} - \int_\gamma \frac{\varphi(w) \, dw}{(w-z_0)^n}$$

$$= \int_\gamma \frac{\varphi(w)}{(w-z)^{n-1}} \cdot \frac{1}{(w-z_0)} \cdot \underbrace{\frac{(w-z_0)}{(w-z)}}_{1 + \frac{(z-z_0)}{(w-z)}} \, dw - \int_\gamma \frac{\varphi(w) \, dw}{(w-z_0)^n}$$

$$= \left(\underbrace{\int_\gamma \frac{\varphi(w) \, dw}{(w-z)^{n-1}(w-z_0)}}_{1} - \int_\gamma \frac{\varphi(w) \, dw}{(w-z_0)^n} \right) + (z-z_0) \underbrace{\int_\gamma \frac{\varphi(w) \, dw}{(w-z)^n (w-z_0)}}_{2}$$

Note, $\lim_{z \to z_0} \frac{①}{z-z_0} = (n-1) f_{n+1}(z_0)$ (By DCT)

$$\int_\gamma \frac{\varphi(w)}{(w-z_0)} \left[\frac{1}{(w-z)^{n-1}} - \frac{1}{(w-z_0)^{n-1}} \right] dw$$

$$= \int_\gamma \frac{\varphi(w)}{(w-z_0)} \left[\frac{(w-z_0)^{n-1} - (w-z)^{n-1}}{(w-z)^{n-1}(w-z_0)^{n-1}} \right] dw$$

$a = w - z_0$, $\quad a^n - b^n = \underbrace{(a-b)}_{(z-z_0)}(a^{n-1} + a^{n-2}b + \dots + ab^{n-2} + b^{n-1})$

$b = w - z$,

$$\lim_{z \to z_0} b = a$$

$$\Rightarrow \lim_{z \to z_0} \frac{a^n - b^n}{a-b} = (n-1) a^{n-1}$$

$\lim_{z \to z_0} \frac{②}{z-z_0} = f_{n+1}(z_0)$

Morera's Thm: If $f: \Omega \to \mathbb{C}$ is cts. s.t $\int_\gamma f = 0$ for every closed path γ, then f is diff.

Pf: Fix $z_0 \in \Omega$ and def.

$$F(z) = \int_\gamma f(w)\, dw \quad \text{where } \gamma(0) = z_0,\ \gamma(1) = z$$

Since for another path $\tilde{\gamma}$ from z to z_0, the closed path $\gamma \oplus (-\tilde{\gamma})$ satisfies $\int_{\gamma \oplus (-\tilde{\gamma})} f(w)\, dw = 0$, we have

$$\int_{\gamma \oplus (-\tilde{\gamma})} f = \int_\gamma f - \int_{\tilde{\gamma}} f = 0$$

$$\Rightarrow \int_\gamma f = \int_{\tilde{\gamma}} f$$

So, F is well-defined

By construction done in the proof of Goursat's thm,
$F'(z) = f(z)$

$\because$ F is diff $\Rightarrow$ f is cts.

$$\Rightarrow \quad f_n(z) = \int \frac{f(w)}{(w-z)^n}\, dw$$

is diff. $\forall\, n \geq 1$ & $f_1(z) = f(z)$

So, f has derivatives of all orders

$\Rightarrow f'$ has derivatives of all orders

$\Rightarrow f$ has derivatives of all orders

Cauchy's Thm

(for an arbitrary open set)

<u>Defn</u>: Recall if γ_1, γ_2 are paths from z, w inside Ω, we say $\gamma_1 \sim \gamma_2$ (wrt Ω) if $\exists\, F : [0,1] \times [0,1] \to \Omega$ cts. s.t $F(0,t) = \gamma_1(t)$ & $F(1,t) = \gamma_2(t)$.

These paths are said to be homotopic.

<u>Defn</u>: Constant path $\gamma : [0,1] \to \Omega$ s.t

$$\gamma(t) = z_0 \quad \forall\, t \in [0,1]$$

Thm: Let Ω be an open set & $f : \Omega \to \mathbb{C}$ be diff.

Then $\displaystyle\int_\gamma f = 0$ for any closed path γ in Ω

s.t $\gamma \sim \mathbb{1}_z$ for some $z \in \Omega$

(const. path)

Lem: Let $\gamma \sim \mathbb{1}_z$ wrt Ω. Then $n(\gamma, p) = 0 \ \forall \ p \in \Omega^c$

Pf: Let $\gamma \sim \mathbb{1}_z$ wrt Ω. Then $\exists \ H : [0,1] \times [0,1] \to \Omega$ s.t

$H(0,t) = \gamma(t)$ &

$H(1,t) = z \ \ \forall \ z \in \Omega$

Assumption: If γ is piecewise diff., then $H(s,t)$ is piecewise diff $\forall \ 0 \le s \le 1$

Using H, create the fx^n,

$$g(s) = \frac{1}{2\pi i} \int_0^1 \frac{\gamma_s'(t)}{\gamma_s(t) - p} \, dt$$

where $\gamma_s(t) = H(s,t)$

winding no. of $\gamma_s(t)$ wrt p

Notice, $g(s)$ is cts. wrt s because $\gamma_s(t) - p \neq 0$ since $p \in \Omega^c$

So, $g(s)$ is cts. & $g(s) \in \mathbb{Z}$.

$\Rightarrow g(s)$ is a const.

$\Rightarrow g(0) = g(1)$

$\Rightarrow n(\gamma, p) = n(\mathbb{1}_z, p) \longrightarrow$

$= 0$

$$n(\mathbb{1}_z, p) = \int_\gamma \frac{dz}{z-p}$$

$$= \int_0^1 \frac{\gamma'(t)\,dt}{\gamma(t) - p}$$

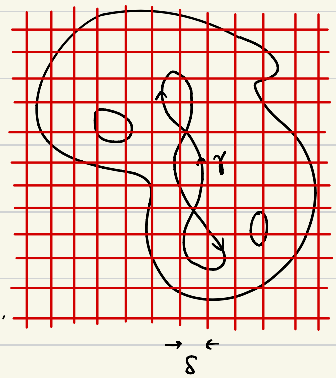
$$\left.\begin{array}{l} \gamma(t) = z \\ \gamma'(t) = 0 \end{array}\right\} = 0$$

Pf: (Cauchy's thm)

Let Ω be bounded.

Ω is bounded $\Rightarrow$ Ω contains finitely many squares.

Let us enumerate the squares by a set J.



$Q_j = j^{th}$ square $j \in J$

Def. $\Gamma_\delta = \sum_{j \in J} \partial Q_j$

Decide ∂Q_j is in the counter-clockwise direction $\forall j$.

$\Omega_\delta = \text{int}\left(\bigcup_{j \in J} Q_j\right)$

Choose $\delta > 0$ so small s.t $\gamma \subseteq \Omega_\delta$

(in particular, $\delta = \dfrac{\text{dist}(\Omega^c, \gamma)}{10^6}$)

Consider $w \in \Omega - \Omega_\delta$

Then $w \in Q$ where $Q \neq Q_j$ for any $j \in J$

Q is a convex set, & since $Q \cap \Omega^c \neq \emptyset$, we have

some $w_0 \in Q \setminus \Omega$

By the lemma, $n(\gamma, w_0) = 0$.

Also, line joining w & w_0 doesn't intersect γ.

$\Rightarrow n(\gamma, w_0) = n(\gamma, w)$

$\Rightarrow n(\gamma, w) = 0 \quad \forall\, w \in Q$

$\Rightarrow n(\gamma, w) = 0 \quad \forall\, w \in \Gamma_\delta$

Choose some $j_0 \in J$. Choose $z \in Q_j$

Then $\displaystyle \frac{1}{2\pi i} \int_{\partial Q_{j_0}} \frac{f(w)}{w - z}\, dw = f(z)$

If $j \neq j_0$, $(z \in Q_{j_0})$

$$\frac{1}{2\pi i} \int_{\partial Q_j} \frac{f(w)}{w - z}\, dw = 0$$

$\Rightarrow \forall z \in Q_{j_0}$

$\Rightarrow f(z) = f(z) + 0 + 0 + \cdots$

$$= \frac{1}{2\pi i} \int_{\partial Q_{j_0}} \frac{f(w)}{w - z}\, dw + \sum_{j \neq j_0} \frac{1}{2\pi i} \int_{\partial Q_j} \frac{f(w)}{w - z}\, dw$$

$$= \sum_{j \in J} \frac{1}{2\pi i} \int_{\partial Q_j} \frac{f(w)}{w - z}\, dw$$

$$= \frac{1}{2\pi i} \int_{\Gamma_\delta} \frac{f(w)}{w - z}\, dw \qquad \forall z \in Q_{j_0}$$

$\Rightarrow$ True $\forall z \in Q_j$, $j \in J$

$\Rightarrow$ True $\forall z \in \bigcup_{j \in J} Q_j = \Omega_\delta + \partial \Omega_\delta$

$$\int_\gamma f(z)\, dz = \int_\gamma \left(\frac{1}{2\pi i} \int_{\Gamma_\delta} \frac{f(w)}{w - z}\, dw \right) dz$$

$$= \int_{\Gamma_\delta} f(w) \left(\frac{1}{2\pi i} \int_\gamma \frac{dz}{w - z} \right) dw$$

$$= - \int_{\Gamma_\delta} f(w) \cdot \underbrace{n(w,\gamma)}_{0} \, dw$$

$$= 0$$

<u>Liouville's thm</u> : $f : \mathbb{C} \to \mathbb{C}$ s.t $f \in C^\infty(\mathbb{C})$ & f is b'nd (meaning $\exists M > 0$) s.t $|f(z)| \leq M \;\; \forall z \in \mathbb{C}$.

Then f = const. fx^n

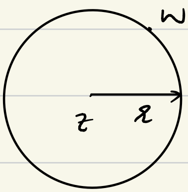
<u>Pf</u> : <u>Idea</u> : Show that $f'(z) = 0 \;\; \forall z \in \mathbb{C}$

$$f'(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(w) \, dw}{(w-z)^2}$$

We show, given $\epsilon > 0$, $|f'(z)| < \epsilon$

$$|f'(z)| \leq \frac{1}{2\pi} \frac{\text{length}(\partial D) \cdot \sup_{w \in \partial D} |f(w)|}{r^2}$$

$$\leq \frac{M}{r}$$



Since $r > 0$ is arbitrary, given $\epsilon > 0$, choose $r = \frac{2M}{\epsilon}$

$\Rightarrow |f'(z)| \leq \epsilon/2 < \epsilon$

Cauchy's estimate: $f^{(n)}(z) = \dfrac{n!}{2\pi i} \displaystyle\int_{\partial D(z,R)} \dfrac{f(w)}{(w-z)^{n+1}} \, dw$

$$|f^{(n)}(z)| \leq \frac{n!}{2\pi} \frac{\sup_{w\in\partial D(z,R)} |f(w)| \cdot 2\pi R}{R^{n+1}}$$

$$= \frac{n!}{R^n} \sup_{w\in\partial D(z,R)} |f(w)|$$

Fundamental thm of algebra

Given a degree $n \geq 1$ polynomial, $f(z)$ has exactly n many roots

i.e $\quad f(z) = a \prod_{i=1}^{n} (z - z_i)$ for some $z_i \in \mathbb{C}$ & $a \in \mathbb{C}$

Pf: Idea: By ind^n & contd^n.

$n=1$: $\quad f(z) = az + b = 0 \Rightarrow z = -b/a$

Let any $n-1$ deg. poly. have $(n-1)$ roots.

Consider an n deg. poly. f.

Notice, if f has at least one root i.e $f(z_1) = 0$, then

$$f(z) = a_0 + a_1 z + \dots + a_n z^n$$

$$0 = f(z_1) = a_0 + a_1 z_1 + \dots + a_n z_1^n$$

Taking diff.,

$$f(z) - f(z_1) = a_1(z - z_1) + a_2(z^2 - z_1^2) + \dots + a_n(z^n - z_1^n)$$

$$\Rightarrow \quad f(z) = (z - z_1) \cdot g(z), \quad \text{where deg } g(z) = n-1.$$

$$(\because z - z_1 \mid z^k - z_1^k \ \forall k \geq 1)$$

But by ind^n, $g(z)$ has $(n-1)$ roots.

So, f has n roots.

<u>To show</u>: f has at least 1 root.

Suppose that f has no roots.

Then $g(z) = \dfrac{1}{f(z)}$ is diff.

Since $f(z)$ is a poly., $|f(z)| \to \infty$ as $|z| \to \infty$

$$|f(z)| = |a_n z^n - (-a_{n-1} z^{n-1} - \dots - a_0)|$$

$$\geq \Big| \underbrace{|a_n z^n|}_{(1)} - \underbrace{|a_{n-1} z^{n-1} + \dots + a_0|}_{(2)} \Big|$$

If $(1) \geq 2 \times (2)$ for some big $|z|$, then $|f(z)| \geq \frac{1}{2}(1) = \frac{1}{2}|a_n||z|^n \to \infty$ as $|z| \to \infty$

<u>Goal</u>: Find $M>0$ s.t when $|z|\geqslant M$, then

$$|a_n||z|^n \geqslant 2|a_{n-1}z^{n-1}+\dots+a_0|$$

$$\begin{aligned}|a_{n-1}z^{n-1}+\dots+a_0| &\leqslant |a_{n-1}||z|^{n-1}+\dots+|a_0| \\ &\leqslant A(|z|^{n-1}+\dots+1) \quad , \quad A=\max_{0\leq k\leq n-1}\{a_k\} \\ &\leqslant nA|z|^{n-1}\end{aligned}$$

Choose $nA|z|^{n-1}\leqslant \frac{1}{2}|a_n||z|^n \Rightarrow |z|\geqslant \frac{2nA}{|a_n|}=M$

$\Rightarrow |f(z)|\geqslant \frac{1}{2}|a_n|\cdot|z|^n$ for $|z|>M=\frac{2nA}{|a_n|}$

i.e $|f(z)|\geqslant \frac{1}{2}|a_n|M^n$ for $|z|>M$

$\Rightarrow |g(z)|=\frac{1}{|f(z)|}\leqslant \frac{2}{|a_n|M^n}$

& $D(0,M)$ is compact, g is cts., so $\exists B>0$ s.t $|g(z)|\leq B$

$\Rightarrow g(z)\leqslant \max\left(B,\frac{2}{|a_n|M^n}\right)$

By Liouville's thm, g is const. $\rightarrow$ Contd^n

($\because f$ is <u>not</u> const.)

Lem: f is diff. on $\Omega' = \Omega \setminus \{a\}$, where Ω = open, $a \in \Omega$.

Then $\exists\, g : \Omega \to \mathbb{C}$ diff. on Ω s.t $f = g$ on Ω' iff

$$\lim_{z \to a} (z-a) f(z) = 0$$

Pf: ($\Rightarrow$) Consider $g : \Omega \to \mathbb{C}$ with g diff. & $g = f$ on Ω'.

Then $$\lim_{z \to a} (z-a) f(z) = \lim_{z \to a} (z-a) g(z) = 0 \cdot g(a) = 0$$

($\because$ g is diff $\Rightarrow$ g is cts.)

($\Leftarrow$) (Application of Cauchy's Integral formula)

Choose $r > 0$ small enough s.t $B(a, r) \subseteq \Omega$ & then

since $\lim_{z \to a} (z-a) f(z) = 0$, we have

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(w)}{w - z} \, dw \qquad \forall\ z \neq a,\ z \in D(a, r)$$

However, $$\frac{1}{2\pi i} \int_{\partial D} \frac{f(w)}{w - z} \, dw = f_1(z) \quad \text{is defined } \forall\ z \in D(a, r)$$

Hence, def. $$g(z) = \begin{cases} \dfrac{1}{2\pi i} \displaystyle\int_{\partial D} \frac{f(w)}{w - z} \, dw, & z = a \\[2ex] f(z), & z \neq a \end{cases}$$

Then $g : \Omega \to \mathbb{C}$ is diff. & $g = f$ on Ω'.

$\underline{\text{Def}^n}$: A pt. $a \in \Omega$ is called a removable singularity of $f : D(a,r) \setminus \{a\} \to \mathbb{C}$ where $r > 0$ is s.t $D(a,r) \subseteq \Omega$ & f is diff. on $D(a,r) \setminus \{a\}$, if $\lim_{z \to a} (z-a) f(z) = 0$

<u>Power series expansion of f in $D(a,r) \subseteq \Omega$</u>

Consider $F(z) = \dfrac{f(z) - f(a)}{z-a}$ defined on $\Omega \setminus \{a\}$.

Let f be diff. on Ω.

Notice, $\lim_{z \to a} (z-a) F(z) = 0$

$\Rightarrow \lim_{z \to a} f(z) - f(a) = 0$

$\Rightarrow$ a is removable singularity of F.

i.e $\exists f_1 : \Omega \to \mathbb{C}$ & $f_1(z) = \dfrac{f(z) - f(a)}{z-a}$ for $z \neq a$, $z \in D(a,r)$.

$\Rightarrow f(z) = f(a) + (z-a) f_1(z)$ for $z \in D(a,r)$

& notice $f_1(a) = \lim_{z \to a} F(z) = f'(a)$

Let's repeat the step, replacing $f(z)$ with $f_1(z)$.

$$F_1(z) = \frac{f_1(z) - f_1(a)}{z-a} \quad \text{for } z \in D(a,r) \setminus \{a\}$$

Again, $\lim_{z \to a} (z-a) F_1(z) = \lim_{z \to a} f_1(z) - f_1(a) = 0$

$\Rightarrow \exists f_2(z) : D(a,r) \to \mathbb{C}$ s.t $f_2(z) = F_1(z)$.

$$f_2(z) = \frac{f_1(z) - f_1(a)}{z-a}$$

$$\Rightarrow f_1(z) = f_1(a) + f_2(z)(z-a)$$

$$\Rightarrow f(z) = f(a) + (z-a) \overbrace{f_1(a)}^{= f'(a)} + (z-a)^2 f_2(z)$$

$\vdots$

$$f(z) = f(a) + (z-a) f_1(a) + (z-a)^2 f_2(a) + \dots + (z-a)^n f_n(z)$$

Diff. n-many times,

$$f^{(n)}(z) = \frac{d^n}{dz^n} ((z-a)^n f(z))$$

Putting $z = a$,

$$f^{(n)}(a) = \left(\frac{d^n}{dz^n} (z-a)^n \right) f_n(a) = n! \, f_n(a)$$

$$\Rightarrow f_n(a) = \frac{1}{n!} f^{(n)}(a)$$

<u>Taylor expansion</u> :

$$f(z) = f(a) + \frac{(z-a)}{1!} f'(a) + \frac{(z-a)^2}{2!} f''(a) + \ldots + \frac{(z-a)^{n+1}}{(n+1)!} f^{(n+1)}(a)$$

$$+ \underbrace{R_n(z)}_{= (z-a)^n f_n(z)}$$

So, $\mathbb{C}$-diff. $\Rightarrow$ Analytic.

Such $f : \Omega \to \mathbb{C}$ are called holomorphic fx^{n}s on Ω.
If $f : \mathbb{C} \to \mathbb{C}$ is diff., then f is called an entire fx^n.

Nature of zeroes of holomorphic fx^ns

Thm: If $f : \Omega \to \mathbb{C}$ is holomp. s.t the zeroes of f have a limit pt, then $f \equiv 0$ on Ω.

Pf: Let $\{w_k\}_{k \geq 1} \subseteq \{z : f(z) = 0\}$, which is Cauchy i.e $\exists\, z \in \Omega$ s.t $\lim_{k \to \infty} w_k = z$. Then $\lim_{k \to \infty} f(w_k) = f(z)$.

Choose $\epsilon > 0$ small enough s.t $B(z, \epsilon) \subseteq \Omega$ & Taylor expand f in $B(z, \epsilon)$.

For $w \in B(z, \epsilon)$

$$f(w) = a_0 + a_1(w-z) + a_2(w-z)^2 + \dots$$

Suppose f is <u>not</u> identically 0 on $B(z, \epsilon)$.

Then $\exists\, n \geq 1$ s.t $a_n \neq 0$.

$$\begin{aligned} f(w) &= a_n(w-z)^n + a_{n+1}(w-z)^{n+1} + \dots \\ &= (w-z)^n \Big[a_n + \underbrace{a_{n+1}(w-z) + \dots}_{\text{Convergent Power Series}}\Big] \\ &= (w-z)^n \left[a_n + g(w)\right] \end{aligned}$$

Plug in $w = w_k$ for k large enough.

Then, $\underbrace{f(w_k)}_{=0} = \underbrace{(w_k - z)}_{\neq 0}{}^n [a_n + g(w_k)] \quad \forall\ k \geq 1$

$\lim_{w_k \to z} g(w_k) = 0$, i.e given any $\epsilon > 0$, $\exists K$ s.t

$k > K \Rightarrow |g(w_k)| < \epsilon$

Take $\epsilon = |a_n|/2$

$\Rightarrow a_n + g(w_k) \neq 0$ for $k > K$ → Contd^n

$$\left[\begin{array}{l} \because\ f(w_k) = 0 \\ \text{but } (w_k - z)^n [a_n + g(w_k)] \neq 0 \end{array}\right]$$

Next, we show $\mathcal{U} = \{z : f(z) = 0\}$ is both open & closed

<u>$\mathcal{U}$ is closed</u>: Take $\{z_k\}_{k \geq 1} \subseteq \mathcal{U}$ a Cauchy seq.

Then $f(z_k) = 0$.

f is holomp $\Rightarrow$ f is cts. $\Rightarrow$ $\underbrace{\lim_{k \to \infty} f(z_k)}_{0} = f(\lim_{k \to \infty} z_k)$

$\Rightarrow \lim_{k \to \infty} z_k \in \mathcal{U}$

$\mathcal{U}$ is open: Consider $z_0 \in \mathcal{U}$.

Already proved that $f \equiv 0$ on $B(z_0, \epsilon) \Rightarrow B(z_0, \epsilon) \subseteq \mathcal{U}$

Result: If $f : \Omega \to \mathbb{C}$ is holomp. & $f \not\equiv 0$, then its zeros are isolated. i.e

given $z \in \Omega$ s.t $f(z) = 0$, $\exists \, \epsilon > 0$ s.t $f(z) = 0$ but $f(w) \neq 0 \;\; \forall \, w \in B(z, \epsilon) \setminus \{z\}$

Let us define the order of a zero of f.

Let $f(z_0)=0$.

$\exists\, r>0$, s.t $\quad f(z)=(z-z_0)^n \cdot g(z) \quad -(1)$

where $g(z)$ is holomp. in $B(z_0,r)$, $g\neq 0$ on $B(z_0,r)$

& (1) is unique.

<u>Pf</u>: Write $f(z)=a_n(z-z_0)^n+a_{n+1}(z-z_0)^{n+1}+\dots$

where $a_n\neq 0$, $n\geq 1$.

$\Rightarrow f(z)=(z-z_0)^n\cdot g(z)$,

$g(z)=a_n+a_{n+1}(z-z_0)+a_{n+2}(z-z_0)^2+\dots \quad$ is holomp.

Choose $r_1\leq r$ s.t $\quad a_{n+1}(z-z_0)+a_{n+2}(z-z_0)^2+\dots < |a_n|/2$

$\forall\, z\in B(z_0,r_1) \Rightarrow g(z)\neq 0 \quad \forall\, z\in B(z_0,r_1)$

& $f(z)=(z-z_0)^n\cdot g(z)$

Here, n is called the order of zero of f.

<u>Ex</u>: n & $g(z)$ are unique

<u>Def^n</u>: Let f be holomp. in $\Omega \setminus \{a_1, a_2, \dots\}$

a_i is called an <u>isolated singularity</u> of f if

$\exists\ V_i \ni a_i$, $V_i \subseteq \Omega \setminus \{a_1, a_2, \dots\}$

If $\lim_{z \to a} (z-a) f(z) = 0$, a is called <u>removable singularity</u>

<u>Def^n</u>: An isolated singularity (at z_0) is called a <u>pole</u> of $f(z)$ if the fn^n $\tilde{f} = \begin{cases} 1/f(z), & \text{if } z \neq z_0 \\ 0, & \text{if } z = z_0 \end{cases}$

is <u>holomp.</u> in some nbd of z_0.

<u>Thm</u>: If f has a pole at z_0, then $f(z) = (z-z_0)^{-n} h(z)$ where $h(z) \neq 0\ \forall z \in U$ for some nbd U of z_0.

This n is <u>unique</u> & is called the order of the pole.

Pf: By def^n, $\tilde{f}$ is holomp. in U.
Then $\tilde{f}(z) = (z-z_0)^n g(z)$ where $g(z) \neq 0 \ \forall z \in U$.

$$\Rightarrow \forall z \in U \setminus \{z_0\}, \quad \tilde{f}(z) = \frac{1}{f(z)}$$

$$\Rightarrow \frac{1}{\tilde{f}(z)} = (z-z_0)^n g(z) \quad \forall z \in U \setminus \{z_0\}$$

$$\Rightarrow f(z) = (z-z_0)^{-n} \underbrace{\frac{1}{g(z)}}_{= h(z) \text{ is holomp.}} \quad \forall z \in U \setminus \{z_0\}$$

Uniqueness of n in the previous result
$\Rightarrow$ uniqueness of n here.

Q. Why is the order of a pole always finite?

A. If f has order n, then in a disc $D(z_0, \epsilon)$, we have

$$f(z) = (z-z_0)^n g(z)$$
$$= (z-z_0)^n \sum_{k=0}^{\infty} a_k (z-z_0)^k$$
$$= \sum_{k=0}^{\infty} a_k (z-z_0)^{n+k} = \sum_{l=0}^{\infty} b_l (z-z_0)^l$$

$\Rightarrow b_0 = b_1 = b_2 = \dots b_{n-1} = 0$

If z_0 has order ∞, then $b_l = 0 \ \forall \ l$

$\Rightarrow f = 0$ on U

$\Rightarrow f = 0$ on Ω (given Ω is connected)

So, order of a pole (from the def^n that $1/f(z)$ is holomp. in a nbd.) is finite.

If $f(z)$ has a pole at $z = z_0$, say order n, then $\lim_{z \to z_0} (z-z_0)^{\alpha} f(z) = 0 \ \forall \ \alpha > n$ because

$f(z) = (z-z_0)^{-n} h(z)$ in a nbd of z_0,

$\Rightarrow \lim_{z \to z_0} (z-z_0)^{\alpha - n} h(z) = 0$

Since n is finite, $\exists$ finite α s.t $\lim_{z \to z_0} (z-z_0)^{\alpha} f(z) = 0$

Does $\exists \alpha > 0$ s.t $\lim_{z \to 0} z^{\alpha} e^{1/z} = 0$

A. No!

$$\lim_{z\to 0} |z^{\alpha} e^{1/z}| = \lim_{z \to 0} |z|^{\alpha} |e^{1/z}|$$

Write $z = re^{i\theta}$, $z \to 0$ means $r \to 0$

$$= \lim_{r \to 0} r^{\alpha} e^{\frac{\cos\theta}{r}} \qquad (\because |e^{z}| = e^{Re(z)})$$

$f(z)$ is said to have an essential singularity at $z = z_0$ if

$$\lim_{z \to z_0} |(z-z_0)^{\alpha} f(z)| = \infty$$

Def^n: f is called a meromorphic fx^n on Ω if f is holomp. on $\Omega \setminus \{a_1, a_2, \dots\}$ s.t each of a_i is a pole.

Let Ω be an open set. Let f be meromorphic on Ω with finitely many poles $a_1, \dots, a_n$.

<u>Not^n</u>: $\Omega' = \Omega \setminus \{a_1, \dots, a_n\}$

<u>Cauchy's Residue Thm</u>: If $\gamma \subseteq \Omega'$ is a closed curve s.t $\gamma \sim \mathbb{1}_{w_0}$ wrt Ω, then $\int_\gamma f(z)\,dz = \sum_{k=1}^{n} n(\gamma, a_k) \cdot \text{Res}(f, a_k)$

<u>Residue of f at z_0</u>:

$$
\begin{aligned}
f(z) &= (z-z_0)^{-n} h(z) \quad \forall z \in U \\
&= (z-z_0)^{-n} \sum_{k=0}^{\infty} a_k (z-z_0)^k \\
&= \sum_{k=0}^{\infty} a_k (z-z_0)^{k-n} \\
&= \frac{a_0}{(z-z_0)^n} + \frac{a_1}{(z-z_0)^{n-1}} + \dots + \underbrace{\frac{a_n}{(z-z_0)}}_{\text{Res}(f, z_0)} + \text{const.} + G(z)
\end{aligned}
$$

Said another way

$$
f(z) = \frac{b_{-n}}{(z-z_0)^n} + \frac{b_{-n+1}}{(z-z_0)^{n+1}} + \dots + \frac{b_{-1}}{(z-z_0)} + b_0 + b_1(z-z_0) + \dots
$$

Then b_{-1} is called the residue of f at z_0 & is denoted by $\text{Res}(f, z_0)$

Recall:

1) If f is holomp. on $D(z_0, r) \Rightarrow \int_\gamma f = 0$

2) If f is holomp. on $R' = R - \{a_1, \dots, a_n\}$ & $\lim_{z \to a_i} (z - a_i) f(z) = 0 \ \forall i$,
then $\int_\gamma f = 0$

3) If f is holomp. on Ω & $\gamma \sim 1\!\!1_{z_0}$, then $\int_\gamma f = 0$

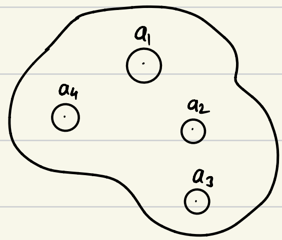
4) $\frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-a} dz = f(a) \cdot n(\gamma, a)$ where $a \in \Omega$, $\gamma \subseteq \Omega \setminus \{a\}$
& f is holomp. on Ω

$$F(z) = \frac{f(z) - f(a)}{z-a} \Rightarrow \int_\gamma F = 0$$

If f is meromophic in Ω s.t f has a single simple pole.

Then $$\frac{1}{2\pi i}\int_\gamma f = \operatorname{Res}(f, a)\cdot n(\gamma, a)$$

We want to generalize this to any meromp. fx^n.



Consider $D(a_1, r_1)$

$$C_1 = \partial D(a_1, r_1) \rightarrow \lambda(t) = a_1 + r_1 e^{it}, \quad 0 \leq t < 2\pi$$

$$f(z) = \frac{b_{-n}}{(z-a_1)^n} + \dots + \frac{b_{-1}}{(z-a_1)} + \underbrace{G(z)}_{\substack{\text{holomp.} \\ \text{in } D(a_1, r_1)}}$$

$$\Rightarrow \int_\gamma f(z)\, dz = \int_\gamma \frac{b_{-n}}{(z-a_1)^n}\, dz + \dots + \int_\gamma \frac{b_{-1}}{(z-a_1)}\, dz + \underbrace{\int_\gamma G(z)\, dz}_{0}$$

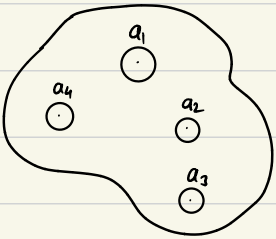
$$\int_\gamma \frac{dz}{z-a_1} = \int_0^{2\pi} \frac{1}{r_1 e^{it}}\, i r_1 e^{it}\, dt = i\int_0^{2\pi} 1\, dt = 2\pi i$$

$$\int_\gamma \frac{1}{(z-a_1)^m}\, dz = \int_0^{2\pi} \frac{1}{(r_1 e^{it})^m}\, i r_1 e^{it}\, dt = \frac{i}{r_1^{m-1}} \int_0^{2\pi} e^{(1-m)it}\, dt = 0$$

$$\Rightarrow \frac{1}{2\pi i}\int_\gamma f(z)\, dz = \operatorname{Res}(f, a_1)$$

Recall: If $\gamma \sim \mathbb{1}_{z_0}$ wrt Ω, then $n(\gamma, a) = 0 \ \forall a \in \mathbb{C} \setminus \Omega$

W/o proof: Converse is true



Let $\tilde{\Omega} = \Omega \setminus \bigcup_{i=1}^{n} D(a_i, r_i)$

Given $\gamma \subseteq \Omega \setminus \{a_1, \dots, a_n\}$

Choose r_i s.t $\gamma \subseteq \tilde{\Omega}$

Consider the 'path' $\gamma - c_1\gamma_1 - c_2\gamma_2 \dots - c_n\gamma_n$

where $c_i = \text{Res}(f, a_i)$ & $\gamma_i = \partial D(a_i, r_i)$

Cauchy's Residue Thm

If $\Omega \subseteq \mathbb{C}$ open, $f : \Omega' \to \mathbb{C}$ is meromorphic with finitely many poles, and $\{\gamma\} \subseteq \Omega$ s.t $\gamma \sim \mathbb{1}_{\mathbb{Z}}$ wrt Ω. (where $\Omega' = \Omega - \{\text{poles of } f\}$), then

$$\frac{1}{2\pi i}\int_\gamma f = \sum_{a \in \text{poles}} n(\gamma, a) \cdot \text{Res}(f, a)$$

$\gamma \sim \mathbb{1}_{\mathbb{Z}}$ wrt Ω iff $n(\gamma, a) = 0 \ \forall a \in \mathbb{C} \setminus \Omega$

$\Rightarrow$ proved

$\Leftarrow$ assumed

Version we'll prove,

If $\Omega \subseteq \mathbb{C}$ open, . . . s.t $n(\gamma, a) = 0 \ \forall a \in \mathbb{C} \setminus \Omega$,

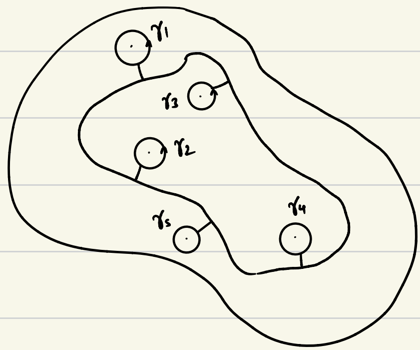
then $$\frac{1}{2\pi i}\int_\gamma f = \sum_{a \in \text{poles}} n(\gamma, a) \cdot \text{Res}(f, a)$$

But shown the original statement if you have just one simple pole.

New path

$\gamma - c_1\gamma_1 - c_2\gamma_2 \dots - c_n\gamma_n = \tilde{\gamma}$

where $c_j = n(\gamma, a_j)$



$$\tilde{\Omega} = \Omega - \bigcup_{j=1}^{n} D(a_j, \tfrac{r_j}{2})$$

Calculate, $n(\tilde{\gamma}, a)$ for $a \in \mathbb{C} \setminus \Omega$

$=$

$$n(\gamma, a) - c_1 n(\gamma_1, a) - c_2 n(\gamma_2, a) \dots - c_n n(\gamma_n, a)$$

To find $n(\gamma_k, a)$ for $a \in D(a_j, r_j/2)$

Pick $a = a_j$

$n(\gamma_j, a_j) = 1$

$n(\gamma_k, a_j) = 0 \quad (j \neq k)$

Now, $n(\tilde{\gamma}, a) = 0 \quad \forall\ a \in \Omega \setminus \tilde{\Omega}$

for $a \in D(a_j, r_j/2)$, $\quad n(\tilde{\gamma}, a) = n(\gamma, a) - c_1 n(\gamma_1, a) - \dots - c_n n(\gamma_n, a)$

$$= \underbrace{n(\gamma, a_j)}_{c_j} - c_j \underbrace{n(\gamma_j, a_j)}_{1}$$

$$= 0$$

So, $\tilde{\gamma} \sim \mathbb{1}_z$ wrt $\tilde{\Omega}$

Since f is holomp. on $\tilde{\Omega}$.

$$\Rightarrow \int_{\tilde{\gamma}} f = 0$$

$$\Rightarrow \int_{\gamma} f - c_1 \int_{\gamma_1} f - c_2 \int_{\gamma_2} f \dots - c_n \int_{\gamma_n} f = 0$$

$$\Rightarrow \frac{1}{2\pi i} \int_{\gamma} f = n(\gamma, a_1) \operatorname{Res}(f, a_1) + n(\gamma, a_2) \operatorname{Res}(f, a_2) + \dots n(\gamma, a_n) \operatorname{Res}(f, a_n)$$

Calculating residues:

If f has a pole of order n at $z = 0$.

$$f(z) = \frac{a_{-n}}{(z-a)^n} + \dots + \frac{a_{-2}}{(z-a)^2} + \frac{a_{-1}}{(z-a)} + a_0 + \dots$$

$$\Rightarrow (z-a)^n f(z) = a_{-n} + a_{-n+1}(z-a) + \dots + a_{-1}(z-a)^{n-1} + a_0(z-a)^n + \dots$$

then $$\operatorname{Res}(f, a) = \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} \left((z-a)^n f(z)\right) \Bigg|_{z=a}$$

Story so far:

1. Complex plane
2. Power series
3. Differentiation
4. Path integrals, Winding no.
5. Cauchy's Residue Thm
6. Applications: Fundamental Thm of Algebra, Morera's thm, etc.
7. Isolated singularities
   - Removable
   - Poles
   - Essential singularities

Laurent expansion

So far, if $f(z)$ has a removable singularity or a pole at $z=a$, then we may write $f(z) = \sum_{k=-n}^{\infty} a_k (z-a)^k$ for $z \in D(a,r)$ & $r>0$ small enough.

$$a_k = \frac{1}{2\pi} \int_{\partial D(a,r)} \frac{f(w)}{(w-a)^{k+1}} \, dw$$

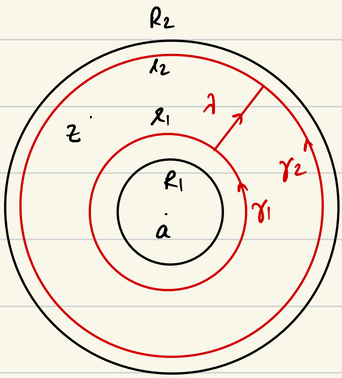
Now, if $z=a$ is an isolated singularity of $f(z)$, then

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z-a)^k$$

for $z \in B(a,R) \setminus \{a\}$.

Moreover, the series conv. absolutely & uniformly on $\mathrm{ann}(a, R_1, R_2)$ where $R > R_2 > R_1 > 0$ $= \{R_1 \le |z-a| \le R_2\}$

Let $\gamma = \gamma_2 - \lambda - \gamma_1 + \lambda \sim 0$ w.r.t $\mathrm{ann}(a, R_1, R_2)$



By Cauchy's Thm,

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w-z} \, dw$$

$$= \underbrace{\frac{1}{2\pi i} \int_{\gamma_2} \frac{f(w)\, dw}{w-z}}_{\substack{\text{holomp for} \\ z \in B(a,R_2)^\circ \\ = F(z) \text{ say}}} - \frac{1}{2\pi i} \int_{\gamma_1} \frac{f(w)\, dw}{w-z}$$

then $F(z) = \sum_{k=0}^{\infty} a_k (z-a)^k$ (Taylor expansion of holomp. fx^n)

Let $G(z) = \frac{-1}{2\pi i} \int_{\gamma_1} \frac{f(w)}{w-z} \, dw$

Let $z' = \frac{1}{z-a}$ where $z \in \mathbb{C} \setminus B(a, r_1)$

$\Rightarrow z \in \mathbb{C} \setminus B(a, r_1) \Leftrightarrow z' \in B(0, 1/r_1)$

$G(z) = G\left(a + \frac{1}{z'}\right) = H(z')$

<u>Claim</u> : $z' = 0$ is a removable singularity of H.

i.e $\lim_{z' \to 0} z' H(z') = 0$

<u>Pf</u> : Instead consider $\lim_{z' \to 0} H(z') = \lim_{|z| \to \infty} G(z)$

$$= \lim_{|z| \to \infty} \frac{-1}{2\pi i} \int_{\gamma_1} \frac{f(w)}{w-z} \, dw$$

$$= 0$$

$$\Rightarrow G(z) = \sum_{k=1}^{\infty} b_k (z')^k$$

$$= \sum_{k=1}^{\infty} \frac{b_k}{(z-a)^k}$$

$$b_k = \frac{1}{2\pi i} \int \frac{H(z')\, dz'}{(z')^{k+1}} \qquad z' = \frac{1}{z-a}$$

$$= \frac{1}{2\pi i} \int_{\gamma_1} \frac{(z-a)^{k+1} G(z)\, dz}{(z-a)^2} \qquad \Rightarrow dz' = \frac{-1}{(z-a)^2}\, dz$$

$$= \frac{1}{2\pi i} \int_{\gamma_1} \frac{G(z)\, dz}{(z-a)^{-k+1}}$$

$$= \frac{1}{2\pi i} \int_{\gamma_1} \frac{f(z) - F(z)\, dz}{(z-a)^{-k+1}}$$

$$= \frac{1}{2\pi i} \int_{\gamma_1} \frac{f(z)\, dz}{(z-a)^{-k+1}} - \underbrace{\frac{1}{2\pi} \int_{\gamma_1} \frac{F(z)\, dz}{(z-a)^{-k+1}}}_{0}$$

$$= a_{-k} \qquad (\because F \text{ is holomp.})$$

This establishes the Laurent expansion.

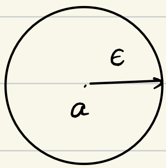
Compute $\lim_{z \to 0} |e^{1/z}| = ?$

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!\, z^n}$$

i.e $z = 0$ is an essential singularity.

Casorati - Weierstrass thm

Given $\epsilon > 0$. If $z = a$ is an essential singularity of f, then $f(D^\times(a, \epsilon))$ is dense in $\mathbb{C}$.



Pf: Suppose $f(D^\times(a, \delta))$ is <u>not</u> dense.

Then $\exists\ c \in \mathbb{C}$ & $\epsilon > 0$ s.t $|f(z) - c| > \epsilon \quad \forall\ z \in D^\times(a, \delta)$

Consider $g(z) = \dfrac{1}{f(z) - c}$, then is holomp. in $D^\times(a, \delta)$

& $|g(z)| < 1/\epsilon$ in $D^\times(a, \delta)$

$\Rightarrow$ $z = a$ is a removable singularity for $g(z)$.

If $g(a) \neq 0$, then $f(a) = c + \dfrac{1}{g(a)}$

$\Rightarrow z = a$ is a removable singularity for $f(z)$ → Contd^n

If $g(a) = 0$, then $z = a$ is a pole of $f(z) - c$ → Contd^n

Next few results :

1. Argument principle : $\displaystyle \frac{1}{2\pi i} \int_{\partial D} \frac{f'(z)}{f(z)}\, dz = $ # zeroes − # poles of f inside D

2. Rouche's thm

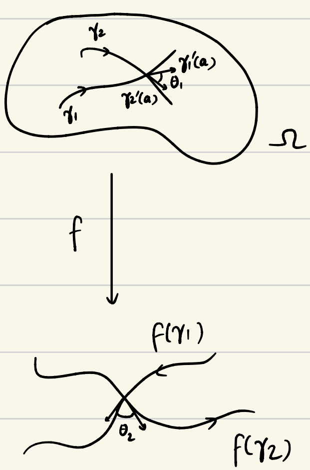
3. Open mapping thm

4. Maximum Modulus principle

Idea of proof (1) ⇒ (2) ⇒ (3) ⇒ (4)

5. Conformal map

$\theta_1 = \theta_2$



Argument principle : Let f be meromorphic in Ω &

γ be a path in Ω that misses poles & zeros of f s.t $\gamma \sim \mathbb{1}_z$ wrt Ω

Then $$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \sum_{\substack{a_j \text{ - distinct} \\ \text{zeroes of } f}} n(\gamma, a_j)\,(\text{order of } a_j) - \sum_{\substack{b_k \text{ - distinct} \\ \text{poles of } f}} n(\gamma, b_k)\,(\text{order of } b_k)$$

Pf : Let a_j be a zero of order u_j.

Then $f(z) = (z-a_j)^{u_j} g(z)$ for some $D(a_j, \epsilon_j)$

$\Rightarrow f'(z) = u_j (z-a_j)^{u_j - 1} g(z) + (z-a_j)^{u_j} g'(z)$

$$\frac{f'(z)}{f(z)} = \frac{u_j}{z-a_j} + \underbrace{\frac{g'(z)}{g(z)}}_{\text{holomp.}}$$

$g'(z)$ ← holomp.

$g(z)$ ← non-vanishing & holomp.

$\Rightarrow z = a_j$ is a simple pole with residue u_j.

Let b_j be a pole of order v_j.

Then $f(z) = \dfrac{g(z)}{(z-b_j)^{v_j}}$ for some $D(b_j, \epsilon_j)$

$$\Rightarrow f'(z) = \frac{(z-b_j)^{v_j} g'(z) - v_j (z-b_j)^{v_j - 1} g(z)}{(z-b_j)^{2v_j}}$$

$$\Rightarrow f'(z) = \frac{g'(z)}{(z-b_j)^{v_j}} - \frac{v_j\, g(z)}{(z-b_j)^{v_j+1}}$$

$$\Rightarrow \frac{f'(z)}{f(z)} = \frac{g'(z)}{g(z)} - \frac{v_j}{(z-b_j)}$$

$\Rightarrow$ $z = b_j$ is a simple pole with residue $-v_j$.

Then apply Residue thm,

$$\frac{1}{2\pi i}\int \frac{f'(z)}{f(z)}\,dz = \sum_{a \in \text{poles of } \frac{f'}{f}} n(\gamma, a) \cdot \operatorname{Res}\left(\frac{f'}{f}, a\right)$$

$$= \sum_{a_j \text{ - distinct zeroes of } f} n(\gamma, a_j)\,(\text{order of } a_j) - \sum_{b_k \text{ - distinct poles of } f} n(\gamma, b_k)\,(\text{order of } b_k)$$

Rouche's Thm

(Motivation)

Let Ω be a b'nd set & $f: \Omega \to \mathbb{C}$ be holomp.

Then the no. of zeroes of f inside Ω is finite.

Thm: Let γ be a Jordan curve (i.e γ is homeomp. to S^1)

s.t $\mathbb{C} \setminus [\gamma]$ has exactly two connected comps.

Let Ω be the b'nd component of $\mathbb{C} \setminus [\gamma]$.

Let $f, g : \Omega \cup [\gamma] \to \mathbb{C}$, holomp. s.t $|f(z)| > |g(z)|$ $\forall z \in \gamma$.

Then $f(z)$ & $f(z)+g(z)$ has the same no. of zeroes inside Ω.

Pf: Def. $f_t(z) = f(z) + tg(z)$ $(0 \le t \le 1)$

Consider n_t = no. of zeroes of f_t inside Ω

Q. Does f_t vanish on γ?

A. Since $|f(z)| > |g(z)|$ $\forall z \in \gamma$

If $f(z)+tg(z) = 0$ for some $t \in [0,1]$

$\Rightarrow |f(z)| = t|g(z)| \to$ Contd^n

$\because |f(z)| > |g(z)|$

Apply the argument principle,

$$n_t = \frac{1}{2\pi i}\int_\gamma \frac{f_t'(z)}{f_t(z)}\,dz = \frac{1}{2\pi i}\int_\gamma \frac{f'(z) + t g'(z)}{f(z) + t g(z)}\,dz$$

<u>Claim</u> : n_t is cts.

<u>Pf</u> : Integrand is cts. wrt z & t.

$\{\gamma\}$ is compact.

Since n_t is integer valued $\Rightarrow$ $n_t = \text{const.}$

$\Rightarrow$ $n_0 = n_1$

$\Rightarrow$ no. of zeroes of f

$=$ no. of zeroes of $f+g$ (inside Ω)

Open Mapping Theorem :

A non-const. holomp. fn is an open map

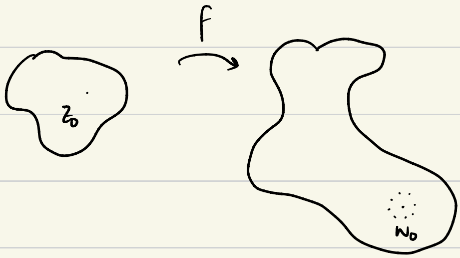
(i.e it maps open sets to open sets)

Pf : Let $f: \Omega \to \mathbb{C}$ be holomp. Let $U \subseteq \Omega$ be open.

To show : $f(U)$ is open

Consider $w_0 \in f(U)$.

Then $\exists z_0 \in U$ s.t $w_0 = f(z_0)$



Goal : $\exists \epsilon > 0$ s.t $B(w_0, \epsilon) \subseteq f(U)$

i.e $\forall w \in B(w_0, \epsilon)$, $\exists z \in U$ s.t $f(z) = w$

Consider $g(z) = f(z) - w$

$$= \underbrace{(f(z) - w_0)}_{F(z)} + \underbrace{(w_0 - w)}_{G(z)}$$

Step 1 : Select $\delta > 0$ s.t $B(z_0, \delta) \subseteq U$, while making sure that $f(z) \neq w_0 \; \forall z \in \partial D(z_0, \delta)$

Since $f(z) \neq w_0 \; \forall z \in \partial D(z_0, \delta)$, choose $\epsilon > 0$ s.t

$|f(z) - w_0| > \epsilon \quad \forall z \in \partial D(z_0, \delta)$

If $|w-w_0|<\epsilon$, then $|F(z)|>|G(z)|$ $\forall z \in \partial D(z_0,\delta)$

Then, $\underbrace{\text{no. of zeroes of } F}_{\text{at least } 1}$ = no. of zeroes of $F+G = f(z)-w$

Maximum Modulus Principle

(for bounded domains)

Let $f:\Omega \to \mathbb{C}$ holomp., Ω - open & b'nd.

If $\exists\, a \in \Omega$ s.t $|f(a)| \geq |f(z)|$ $\forall z \in \Omega$, then f is a const.

Pf: Suppose $\exists\, a \in \Omega$ s.t $|f(a)| \geq |f(z)|$ $\forall z \in \Omega$

Choose $r>0$ s.t $B(a,r) \subseteq \Omega$

By Cauchy's Integral formula, $f(a) = \dfrac{1}{2\pi i}\displaystyle\int_{\partial B} \dfrac{f(z)}{(z-a)}\,dz$

$\partial B = a + re^{it}$, $0 \leq t \leq 2\pi$

$$\Rightarrow f(a) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(a+re^{it})}{re^{it}} \cdot ire^{it}\,dt = \frac{1}{2\pi}\int_0^{2\pi} f(a+re^{it})\,dt$$

$$|f(a)| \leq \frac{1}{2\pi}\int_0^{2\pi} \underbrace{|f(a+re^{it})|}_{\leq |f(a)|}\, dt$$

$$\leq |f(a)|$$

$$\Rightarrow \frac{1}{2\pi}\int_0^{2\pi} |f(a)|\, dt = \frac{1}{2\pi}\int_0^{2\pi} |f(a+re^{it})|\, dt$$

$$\Rightarrow \frac{1}{2\pi}\int_0^{2\pi} \underbrace{[\,|f(a)| - |f(a+re^{it})|\,]}_{\geq 0}\, dt = 0$$

$$\Rightarrow |f(a)| - |f(a+re^{it})| = 0 \quad \forall\ 0 \leq t \leq 2\pi$$

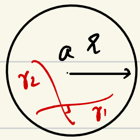
i.e $|f(z)|$ is const. on $\partial B(a,r)$

$\because$ Cauchy's formula is true $\forall\ r_1 \leq r$

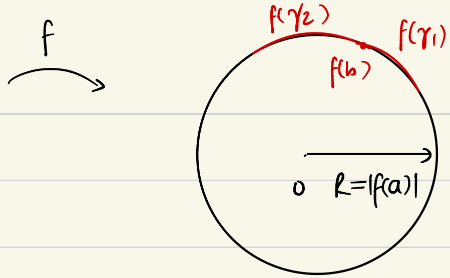
$$\Rightarrow |f(a)| = |f(a+r_1e^{it})| \quad \forall\ 0 \leq t \leq 2\pi,\ 0 < r_1 \leq r$$

This means $\forall\ z \in B(a,r)$, $|f(z)| = |f(a)|$ $\rightarrow$ Contd^n to open mapping thm

($\because$ image is closed set)



angle = 90°
b/w γ_1 & γ_2



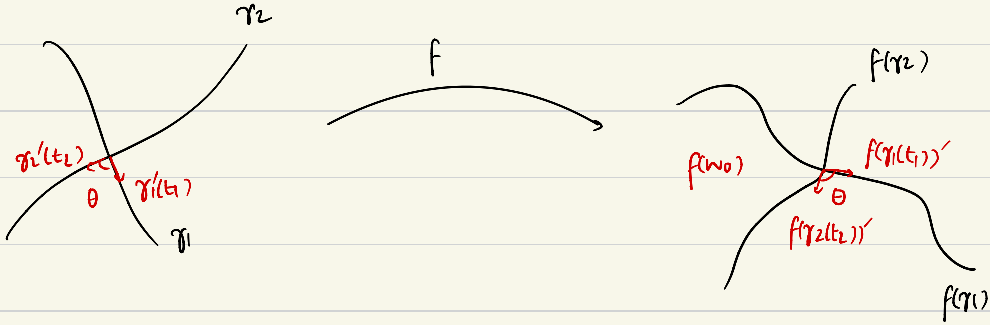
angle b/w
$f(\gamma_1)$ & $f(\gamma_2)$
$= 0°$ or $180°$

Let a be s.t $|f(a)| \geqslant |f(z)| \ \forall z \in \Omega$

So, angle b/w γ_1 & γ_2 is <u>not</u> preserved.

<u>Conformal map</u> : A map $f : \Omega \to \mathbb{C}$ is called conformal if given two paths $\gamma_1, \gamma_2 \subseteq \Omega$ that intersect at $w_0 = \gamma_1(t_1) = \gamma_2(t_2)$ (where $f'(w_0) \neq 0$), then

$$\operatorname{Arg}(\gamma_1'(t_1)) - \operatorname{Arg}(\gamma_2'(t_2)) = \operatorname{Arg}((f(\gamma_1(t_1)))') - \operatorname{Arg}((f(\gamma_2(t_2)))')$$



$f(\gamma_1(t_1))' = f_1'(\gamma_1(t_1)) \cdot \gamma_1'(t_1)$ if $\gamma_1'(t_1) \neq 0$, then

1. $\mathrm{Arg}(f(\gamma_1(t_1))') = \mathrm{Arg}(f'(\gamma_2(t_2))) + \mathrm{Arg}(\gamma_1'(t_1)) + 2\pi k$

$(k = 0, -1)$

2. $\mathrm{Arg}(f(\gamma_2(t_2))') = \mathrm{Arg}(f'(\gamma_1(t_1))) + 2\pi l$

(1) − (2) : $\arg(f(\gamma_1(t_1))') - \arg(f(\gamma_2(t_2))') = \arg(\gamma_1'(t_1)) - \arg(\gamma_2'(t_2))$

Path integral calculation

- Residue thm
- Argument principle

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \underbrace{\lim_{\substack{T_1 \to \infty \\ T_2 \to \infty}}}_{\text{Why does the lim. exist?}} \int_{-T_1}^{T_2} \frac{dx}{1+x^2}$$

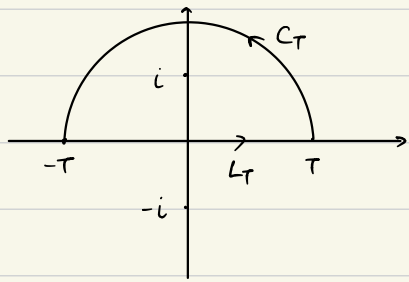
Remainder − $\underbrace{\int_{-\infty}^{-T_1} \frac{1}{1+x^2}\,dx}_{\leq -\frac{2}{T_1}} + \underbrace{\int_{T_2}^{\infty} \frac{1}{1+x^2}\,dx}_{\leq \frac{2}{T_2}} \to 0$ as $T_1, T_2 \to \infty$

$$f(x) = \frac{1}{1+x^2}$$

Instead, consider $f(z) = \dfrac{1}{1+z^2}$ ← 2 poles $1+z^2=0 \Rightarrow z = \pm i$

Interpret $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$ as $\displaystyle\lim_{T\to\infty} \int_{-T}^{T} \frac{dx}{1+x^2}$

$$\int_{-T}^{T} \frac{dx}{1+x^2} = \int_{L_T} \frac{dz}{1+z^2} + \int_{C_T} \frac{dz}{1+z^2} - \int_{C_T} \frac{dz}{1+z^2}$$



$$= \underbrace{\int_{\gamma_T} \frac{dz}{1+z^2}}_{} - \int_{C_T} \frac{dz}{1+z^2}$$

$$2\pi i \left(\mathrm{Res}(f,i) \cdot \underbrace{n(\gamma,i)}_{1} + \mathrm{Res}(f,-i) \cdot \underbrace{n(\gamma,-i)}_{0} \right)$$

$$\frac{1}{z^2+1} = \frac{1}{(z-i)(z+i)}$$

$$\mathrm{Res}(f,i) = \lim_{z\to i} (z-i)\, f(z) = \lim_{z \to i} \frac{z-i}{(z-i)(z+i)} = \frac{1}{2i}$$

$$\int_{\gamma_T} \frac{1}{1+z^2}\, dz = 2\pi i \cdot \frac{1}{2i} = \pi$$

Now, estimate,
$$\int_{C_T} \frac{1}{1+z^2}\, dz$$
$$= \int_0^{\pi} \frac{1}{1+T^2e^{2it}} \cdot Ti\, e^{it}\, dt \qquad z = Te^{it}$$
$$dz = Tie^{it}\, dt$$

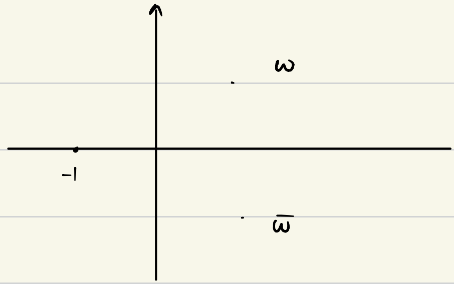
$$\left| \int_{C_T} \frac{dz}{1+z^2} \right| \leq \int_0^{\pi} \frac{T}{|1+T^2e^{2it}|}\, dt \leq \int_0^{\pi} \frac{T}{T^2-1}\, dt \quad \left(\because \ |1+T^2e^{2it}| \geq \big||T^2e^{2it}| - |1|\big| = T^2-1 \right)$$

$$\leq \frac{\pi T}{T^2-1} \to 0 \quad \text{as } T \to \infty$$

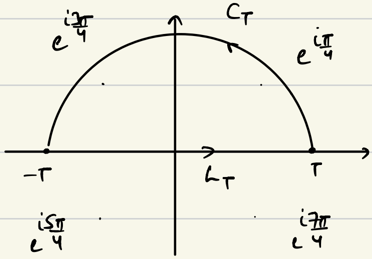
$$\Rightarrow \lim_{T\to\infty} \int_{-T}^{T} \frac{dx}{1+x^2} = \pi + \underbrace{\lim_{T\to\infty} \int_{C_T} \frac{dz}{1+z^2}}_{=0}$$

$$= \pi$$

$$\int_0^\infty \frac{dx}{1+x^3}$$ can be calculated using



$$\int_{-\infty}^{\infty} \frac{dx}{1+x^4}$$



<u>Poles</u> : $z = e^{\frac{i\pi}{4}}, e^{\frac{i3\pi}{4}}, e^{\frac{i5\pi}{4}}, e^{\frac{i7\pi}{4}}$

$$\int_{-T}^{T} \frac{dx}{1+x^4} = \underbrace{\int_{L_T} \frac{dz}{1+z^4} + \int_{C_T} \frac{dz}{1+z^4}}_{\displaystyle\int_{\gamma_T} \frac{dz}{1+z^4}} - \int_{C_T} \frac{dz}{1+z^4}$$

$$\int_{\gamma_T} \frac{dz}{1+z^4} = 2\pi i \left[\operatorname{Res}(f, e^{i\pi/4}) + \operatorname{Res}(f, e^{\frac{i3\pi}{4}}) \right]$$

$$f(z) = \frac{1}{p(z)}$$

$$p(z) = z^4 + 1 = (z - w_1)(z - w_2)(z - w_3)(z - w_4) \quad , \qquad w_k = e^{i(2k-1)\frac{\pi}{4}}$$

$$p'(z) = (z - w_2)(z - w_3)(z - w_4) + (z - w_1)(z - w_3)(z - w_4) + (z - w_1)(z - w_2)(z - w_4)$$
$$+ (z - w_1)(z - w_2)(z - w_3)$$

$$p'(w_1) = (w_1 - w_2)(w_1 - w_3)(w_1 - w_4) = 4w_1^3$$

$$\mathrm{Res}(f, e^{\frac{i\pi}{4}}) = \frac{1}{p'(w_1)} = \frac{1}{4e^{i3\pi/4}}$$

$$\mathrm{Res}(f, e^{i\frac{3\pi}{4}}) = \frac{1}{p'(w_2)} = \frac{1}{4e^{i9\pi/4}} = \frac{1}{4e^{i\pi/4}}$$

$$\mathrm{Res}(f, e^{i\pi/4}) + \mathrm{Res}(f, e^{i\frac{3\pi}{4}}) = \frac{1}{4}\Big(\underbrace{e^{-\frac{3\pi i}{4}}}_{-e^{\frac{i\pi}{4}}} + e^{-\frac{i\pi}{4}}\Big)$$

$$= \frac{-2\sin(\pi/4)\,i}{4} = \frac{-i}{2\sqrt{2}}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{dx}{1+x^4} = 2\pi i\left[\mathrm{Res}(f, e^{i\pi/4}) + \mathrm{Res}(f, e^{i\frac{3\pi}{4}})\right] = \frac{\pi}{\sqrt{2}}$$

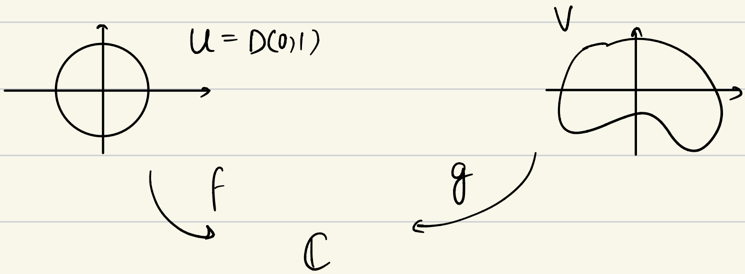
Path integrals : $\displaystyle\int_{\gamma} f(z)\,dz$

γ parametrization

$\gamma_2 : [0,1] \to \mathbb{C}$

$\gamma_1 : [0, 2\pi] \to \mathbb{C}$

Q. Given two sets $U, V \subseteq \mathbb{C}$. Are they the 'same'?



$\mathcal{F}(U) = \{ f : U \to \mathbb{C} \text{ holomp.} \}$

$\mathcal{F}(V) = \{ f : V \to \mathbb{C} \text{ holomp.} \}$

Can we find $\pi : U \to V$ holomp. s.t

Given $f : U \to \mathbb{C}$, we can find $g : V \to \mathbb{C}$ s.t

$f(z) = g(\pi(z))$

$$\begin{tikzcd} U \arrow[r, "\pi"] \arrow[d, "f"'] & V \arrow[dl, "g"] \\ \mathbb{C} & \end{tikzcd}$$

π is holomp. (on U) & π is inv'ble with π^{-1} holomp (on V)

Riemann Mapping Thm

If $S \subset \mathbb{C}$ is a non-trivial (i.e $S \neq \emptyset, \mathbb{C}$) simply conn. open set, then S is biholomorphic to the open unit disc.

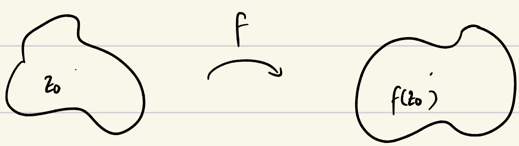
Def^n: Given $U, V \subseteq \mathbb{C}$, we say that $f: U \to V$ is a biholomorphism if f is holomorphic & bijective

Pp^n: If $f: U \to V$ is holomp. & bij, then the inverse map $f^{-1}: V \to U$ is also holomp. (& bij.)

Pf: Prove it via proving if $f: U \to V$ is holomp. & bij., then $f'(z) \neq 0 \ \forall z \in U$

$$\left(\frac{d}{dz} f^{-1}(z) \Big|_{z=z_0} = \frac{1}{f'(f^{-1}(z_0))} \right)$$

Suppose otherwise that $\exists z_0 \in U$ s.t $f'(z_0) = 0$



Idea :

If $f'(z_0) = 0$

$\Rightarrow f(z) = f(z_0) + a_1 \underbrace{f'(z_0)}_{0}(z - z_0) + a_2 f''(z_0)(z-z_0)^2 + \dots$

$\Rightarrow f(z) - f(z_0) = a_2 f''(z_0)(z-z_0)^2 + a_3 f'''(z_0)(z-z_0)^3 + \dots$

$f(z) - f(z_0) = h(z)$ has a zero of order ≥ 2 at z_0

Instead if we consider $f(z) - f(z_0) - w$ where $|w|$ is very small.

Suppose the roots at z_0 move slightly & become z_1 & z_2

Then $f(z_1) - f(z_0) - w = f(z_2) - f(z_0) - w = 0$

$\Rightarrow f(z_1) = f(z_2) \rightarrow$ contdⁿ ($\because f$ was inj.)

$f(z) = f(z_0) + a(z-z_0)^k + G(z)$ where $a \in \mathbb{C}$, $a \neq 0$, $k \geq 2$ &

$\dfrac{G(z)}{(z-z_0)^{k+1}}$ is holomp.

Write $f(z) - f(z_0) - w = F(z) + G(z)$, where $F(z) = a(z-z_0)^k - w$

Want to apply Rouche's thm.

Notice $|f(z)| > |G(z)|$ for $z \in \partial B(z_0, \epsilon)$

$|G(z)| \le C_1 \epsilon^{k+1}$ for $z \in \partial B(z_0, \epsilon)$

$|f(z)| \underset{\substack{\uparrow \\ w=0}}{=} |a|\epsilon^k > |G(z)|$ for small enough

If $|w|$ is small & non-zero, we may choose w, s.t

$|f(z)| > |G(z)|$ for $z \in \partial B(z_0, \epsilon)$

\# zeroes of $f(z)$ is $a(z-z_0)^k - w = 0$

$\Rightarrow (z-z_0)^k = \dfrac{w}{a}$

\# of sol^{n}s $= k \geq 2$

$\Rightarrow$ $f+G$ also has two solutions

Why are the sol^{n}s diff. ?

Because $|w|$ is small, $z = z_0 + \left(\frac{w}{a}\right)^{1/k}$ $\rightarrow$ Contd^n to injectivity

Two interesting simply conn. open sets

1) $\mathbb{H}$ = Upper half plane

$= \{x+iy : x \in \mathbb{R}, y \in \mathbb{R}_{>0}\}$

2) $\mathbb{D}$ = Open unit disc

$= \{z \in \mathbb{C} : |z| < 1\}$

$$F(z) = \frac{i-z}{i+z}$$

Claim: $F : \mathbb{H} \to \mathbb{D}$ is a bihola.

& $G : \mathbb{D} \to \mathbb{H}$, $G(z) = i\left(\frac{1-z}{1+z}\right)$ is the inverse of $F(z)$

$$|F(z)| = \frac{|i-z|}{|i+z|} < 1 \quad (\because z \in \mathbb{H})$$

$\Rightarrow F(z) \in \mathbb{D} \Rightarrow F(\mathbb{H}) \subseteq \mathbb{D}$

Consider $|z| < 1 \Rightarrow x^2 + y^2 < 1$ for $z = x + iy$

$$G(z) = \frac{i(1-x-iy)}{(1+x+iy)} = \frac{i(1-x-iy)(1+x-iy)}{(1+x)^2+y^2} = \frac{2y + (1-x^2-y^2)i}{(1+x)^2+y^2}$$

$\because\ x^2+y^2<1 \Rightarrow \dfrac{1-x^2-y^2}{(1+x)^2+y^2} > 0 \Rightarrow \operatorname{Im}(G(z)) > 0$

$\Rightarrow G(z) \in \mathbb{H}$

$\Rightarrow G(\mathbb{D}) \subseteq \mathbb{H}$

$$F\circ G(z) = F(G(z)) = F\left(i\left(\frac{1-z}{1+z}\right)\right) = \frac{i - i\left(\frac{1-z}{1+z}\right)}{i + i\left(\frac{1-z}{1+z}\right)}$$

$$= \frac{1+z-1+z}{1+z+1-z} = z = Id_{\mathbb{D}}(z)$$

$$G\circ F(z) = G(F(z)) = G\left(\frac{i-z}{i+z}\right) = \frac{i\left(1-\frac{i-z}{i+z}\right)}{\left(1+\frac{i-z}{i+z}\right)} = \frac{i(i+z-i+z)}{i+z+i-z}$$

$$= z = Id_{\mathbb{H}}(z)$$

Find all the biholomp. maps $f: \mathbb{D} \to \mathbb{D}$

Result: $f(z) = e^{i\theta}\left(\dfrac{\alpha - z}{1-\bar{\alpha}z}\right)$, where $\theta \in \mathbb{R}$, $\alpha \in \mathbb{D}$.

T'fore we can find all biholomps.

$$\begin{array}{ccc} g: \mathbb{H} & \longrightarrow & \mathbb{H} \\ \downarrow F & & \uparrow G \\ \mathbb{D} & \xrightarrow{f} & \mathbb{D} \end{array}$$

Not^n: $\mathrm{Aut}(\mathbb{D}) = \{\text{bihlomp } f : \mathbb{D} \to \mathbb{D}\}$

$\mathrm{Aut}(\mathbb{H}) = \{\text{bihlomp } f : \mathbb{H} \to \mathbb{H}\}$

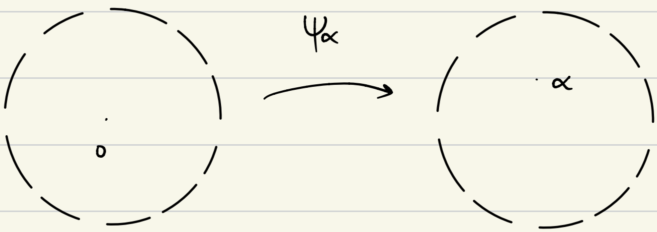
For not^n, $\varphi_\alpha(z) = \dfrac{\alpha - z}{1 - \bar{\alpha} z}$ $\quad (|\alpha| < 1)$

Check that $\varphi_\alpha : \mathbb{D} \to \mathbb{D}$ is a biholomp.

φ_α is diff. if $1 - \bar{\alpha} z \neq 0 \Rightarrow \bar{\alpha} z \neq 1$

$\because |\alpha| < 1$ & $|z| < 1 \Rightarrow |\bar{\alpha} z| < 1 \Rightarrow \varphi_\alpha$ is holomp. on $\mathbb{D}$.

$$\varphi_\alpha(\varphi_\alpha(z)) = \frac{\alpha - \varphi_\alpha(z)}{1 - \bar{\alpha}\varphi_\alpha(z)} = \frac{\alpha - \frac{\alpha - z}{1 - \bar{\alpha} z}}{1 - \bar{\alpha} \cdot \frac{\alpha - z}{1 - \bar{\alpha} z}} = \frac{\cancel{\alpha} - |\alpha|^2 z - \cancel{\alpha} + z}{1 - \cancel{\bar{\alpha} z} - |\alpha|^2 + \cancel{\bar{\alpha} z}} = z$$



$\varphi_\alpha(0) = \alpha$

$\varphi_\alpha(\alpha) = 0$

$\lambda_\theta(z) = e^{i\theta} z$

Then $\lambda_\theta : \mathbb{D} \to \mathbb{D}$ is an automp.

$$\frac{d\lambda_\theta}{dz} = e^{i\theta}$$

$\lambda_\theta^{-1}(z) = \lambda_{-\theta}(z)$

<u>Schwarz lemma</u> : Let $f : \mathbb{D} \to \mathbb{D}$ be holomp. with $f(0) = 0$

Then
1. $|f(z)| \leq |z|$
2. If $\exists z_0 \neq 0$ s.t $|f(z_0)| = |z_0|$, then f is a rot^n
3. $|f'(0)| \leq 1$ with equality iff f is a rot^n.

<u>Pf</u> : (Power series expansion + Maximum Modulus principle)

1) $f(z) = a_1 z + a_2 z^2 + \ldots \Rightarrow \dfrac{f(z)}{z}$ is holomp.

Choose $z \in \mathbb{D}$. Let $r = |z|$

So, $z \in \partial B(0, r)$

$$\left|\frac{f(z)}{z}\right| = \frac{|f(z)|}{|z|} < \frac{1}{r}$$

Also, $f(z)/z$ is holomp. inside $\overline{B(0,r)}$

By max. modulus principle, $\left|\frac{f(z)}{z}\right| \leq \frac{1}{r}$ $\forall\ z \in \overline{B(0,r)}$

Taking $\lim_{r \to 1}$, $\left|\frac{f(z)}{z}\right| \leqslant 1$ $\forall\ z \in \bigcup_{0<r<1} \overline{B(0,r)} = \mathbb{D}$

$$\Rightarrow |f(z)| \leq |z| \quad \forall\ z \in \mathbb{D}$$

2) If $z_0 \neq 0$ & $|f(z_0)| = |z_0|$, then $\left|\frac{f(z_0)}{z_0}\right| = 1$ for $z \in \mathbb{D}$

By max mod principle, $\frac{f(z)}{z} = C$ (const.) with $|C| = 1$

$\Rightarrow C = e^{i\theta}$ for some $\theta \in \mathbb{R}$

$$\Rightarrow f(z) = e^{i\theta} z$$

3) $f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z - 0} = \lim_{z \to 0} \frac{f(z)}{z}$

Since $\left|\frac{f(z)}{z}\right| \leqslant 1$ $\forall\ z \in \mathbb{D}$ $\Rightarrow |f'(0)| \leqslant 1$

On the other hand, if $f(z) = e^{i\theta} z$, then $|f'(z)| = |e^{i\theta}| = 1$

& if $|f'(0)| = 1 \Rightarrow$ By max mod principle, $f(z) = cz$ for $|c| = 1$

$\underline{\text{Ppn}}$: $f : \mathbb{D} \to \mathbb{D}$ is biholomp. $\Rightarrow$ $f(z) = e^{i\theta} \psi_\alpha(z)$ for some $\theta \in \mathbb{R}$, $|\alpha| < 1$

for $\psi_\alpha(z) = \dfrac{\alpha - z}{1 - \bar{\alpha} z}$

$\underline{\text{Pf}}$: Consider biholomp $f : \mathbb{D} \to \mathbb{D}$

Let $\alpha = f(0)$.

$$0 \xmapsto{f} \alpha \xmapsto{\psi_\alpha} 0$$

$g = \psi_\alpha \circ f$, then g satisfies the hypothesis of Schwarz lemma

$\Rightarrow |g(z)| \leq |z|$

$\Rightarrow |g^{-1}(g(z))| \leq |g^{-1}(z)|$

$\Rightarrow |z| \leq |g^{-1}(z)|$

But by Schwarz lemma on $g^{-1}(z)$, $|g^{-1}(z)| \leq |z|$

$\Rightarrow |g^{-1}(z)| = |z|$

$\Rightarrow |g(z)| = |z|$

$\Rightarrow \exists\, \theta \in \mathbb{R}$ s.t $g(z) = e^{i\theta} z$ $\Rightarrow \psi_\alpha \circ f(z) = e^{i\theta} z$

$\Rightarrow f(z) = e^{i\theta} \psi_\alpha(z)$

for $K \subseteq \mathbb{C}$, f is uniformly cts. on K if given any $\epsilon > 0$,

$\exists\, \delta_{\epsilon,f} > 0$ s.t $\forall\, x, y \in K$, $|x-y| < \delta_{\epsilon,f} \Rightarrow |f(x)-f(y)| < \epsilon$

Def^n: Let $\mathcal{F}$ = a family or seq. of fn^{n}s

$\mathcal{F}$ is called equicontinuous on K if given $\epsilon > 0$, $\exists\, \delta_{\epsilon,K} > 0$

s.t $\forall\, x, y \in K$, $|x-y| < \delta_{\epsilon,K} \Rightarrow |f(x)-f(y)| < \epsilon \quad \forall\, f \in \mathcal{F}$

Def^n: (Uniform boundedness)

$\mathcal{F}$ is called uniformly bounded (on compact subsets) if

$\forall\, K \subseteq \text{domain}(\mathcal{F})$, $\exists\, B_K > 0$ s.t $|f(z)| \leq B_K \quad \forall\, z \in K, \forall\, f \in \mathcal{F}$

Def^n: (Normal family)

$\mathcal{F}$ is called normal if every seq. in $\mathcal{F}$ has a subseq. which converges uniformly on every compact set of domain($\mathcal{F}$)

(Limit fn^n need not be in $\mathcal{F}$)

Montel's thm

Let $\mathcal{F}$ be a set of holomp. fx^{n}s from $\Omega \to \mathbb{C}$ (Ω open)

s.t $\mathcal{F}$ is uniformly bounded on compact sets.

Then,

1. $\mathcal{F}$ is equicontinuous $\forall$ compact $K \subseteq \Omega$
2. $\mathcal{F}$ is a normal family

Idea: 1) Use Cauchy Integral Formula

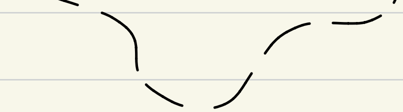
2) Diagonalization argument for subseq. of seq.

Pf: 1) Given $\epsilon > 0$, find $\delta_{\epsilon, K} > 0$

Since Ω is open & K is closed,

choose $r > 0$ small enough s.t

$\overline{B(z, 3r)} \subseteq \Omega$ $\forall z \in K$.



Choose $w \in B(z, r) \cap K$.

If we take $\gamma = \partial B(z, 2r)$

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} \, d\zeta$$

$$f(w) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - w} \, d\zeta$$

$$|f(z) - f(w)| = \frac{1}{2\pi} \left| \int_\gamma f(\zeta) \left(\frac{1}{\zeta - z} - \frac{1}{\zeta - w} \right) d\zeta \right|$$

$$= \frac{1}{2\pi} \left| \int_\gamma f(\zeta) \frac{(z-w)}{(\zeta - z)(\zeta - w)} d\zeta \right|$$

$$\leq \frac{1}{2\pi} \frac{\sup_{\zeta \in \gamma} |f(\zeta)| \cdot |z-w| \quad 2\pi \cdot 2r}{r^2}$$

$$\leq \underbrace{\frac{2B_K}{r}}_{C_K} |z-w|$$

Given $\epsilon > 0$, choose $\delta = \epsilon / C_K$

2) Let compact $K \subseteq \Omega$ & $\{f_n\}_{n \geq 1}$ be a seq. of f^n's in $\mathcal{F}$.
Start with choosing a seq. of pts. in Ω say $\{w_n\}_{n\geq 1}$, s.t
$\{w_n\}_{n \geq 1} \underset{\text{dense}}{\subseteq} \Omega$.

Now, $\mathcal{F}$ is uniformly b'nd on $K \Rightarrow \mathcal{G}_0 = \{f_n\}_{n \geq 1}$ is also uniformly b'nd on K

Consider the seq. of pts $\{f_n(w_1)\}_{n \geq 1}$
Bolzano-Weierstrass gives a conv. subseq. of $\mathbb{C}$-nos.
$\{f_{1,1}(w_1), f_{1,2}(w_1), f_{1,3}(w_1), \dots\}$

Consider the fn's $\mathcal{G}_1 = \{f_{1,1}, f_{1,2}, f_{1,3}, \dots\}$

Evaluate these on w_2, $\{f_{1,1}(w_2), f_{1,2}(w_2), f_{1,3}(w_2), \dots\}$

Again, by BW, we obtain a conv. subseq. $\{f_{2,1}(w_2), f_{2,2}(w_2), f_{2,3}(w_2), \dots\}$

Repeating the process by evaluating $\mathcal{G}_2 = \{f_{2,1}, f_{2,2}, f_{2,3}\}$ at w_3.

This gives us a seq. of sets $\mathcal{G}_0 \supseteq \mathcal{G}_1 \supseteq \mathcal{G}_2 \supseteq \dots$

At jth step, $\{f_{j,n}(w_j)\}_{n\geq 1}$ conv.

Let $g_n = f_{n,n}$. Then $\{g_n(w_j)\}_{n\geq 1}$ conv. $\forall\, j \geq 1$

We proved $\mathcal{F}$ is equicontinuous. So given $\epsilon > 0$, choose $\delta_{\epsilon,K} > 0$ s.t

$|z-w| < \delta_{\epsilon,K} = \delta \Rightarrow |f(z) - f(w)| < \epsilon \quad \forall\, f \in \mathcal{F}$.

Choose an open cover of K, $\{B(w_i, \delta)\}_{i=1}^{\infty}$

Since K is compact, $\exists$ finite subcover $\{B(w_i, \delta)\}_{i=1}^{J}$

Pick $N > 1$ large s.t $n, m > N \Rightarrow |g_m(w_j) - g_n(w_j)| < \epsilon \quad \forall\, j = 1, 2, \dots, J$

If $z \in K$, then $z \in D(w_j, \delta)$ for some $j \in \{1, 2, \dots, j\}$ s.t

$$|g_n(z) - g_m(z)| \leq |g_n(z) - g_n(w_j)| + |g_n(w_j) - g_m(w_j)| + |g_m(w_j) - g_m(z)|$$
$$< 3\epsilon$$

$\Rightarrow \{g_n\}$ conv. uniformly on K.

Finally, make the conv. indep. of K.

Consider $K_\ell = \{ z \in \Omega : \text{dist}(z, \partial\Omega) \geq \frac{1}{\ell} \} \cap \overline{B(0,\ell)}$

Note, $K_1 \subseteq K_2 \subseteq K_3 \subseteq \ldots$ & $\bigcup_{\ell=1}^{\infty} K_\ell = \Omega$

Start with $K = K_1$

$\{g_n\}$ is uniformly conv. on K_1.

Do the same trick with the original seq. $\{f_n\}$ replaced with $\{g_n\}$ & for K_2 to obtain $\{g_{2,1}, g_{2,2}, g_{2,3}, \ldots\} = \{g_{2,n}\}_{n \geq 1}$ conv. uniformly on K.

Repeat the procedure to obtain $\{g_{\ell,n}\}_{n\geq 1}$ conv. uniformly on K_ℓ.

& choose the subseq. $\{g_{n,n}\}_{n \geq 1}$ conv. uniformly $\forall$ compact $K \subseteq \Omega$.

Riemann Mapping thm

Let $\Omega \subseteq \mathbb{C}$ open, proper & simply conn.

Then $\exists$ unique $f : \Omega \to \mathbb{D}$ biholomp. s.t if $f(z_0) = 0$, then $f'(z_0) > 0$.

Rem: $\Omega \neq \emptyset$ is trivial

$\Omega = \mathbb{C}$, then by max-mod. principle, $f = $ const.

Lem 1: Let $\Omega \subseteq \mathbb{C}$ open. Let $\{g_n\}_{n \geq 1}$ be a seq. of holomp. fn's $g_n : \Omega \to \mathbb{C}$ s.t $g_n \to g$ uniformly on any $K \subseteq \Omega$ compact.

Then g is holomp. & $\{g_n'\} \to g'$ uniformly on any $K \subseteq \Omega$ compact.

Pf: Apply Morera's thm.

Let γ be a closed path in Ω s.t $\gamma \sim 1\!\!1_z$ wrt Ω.

Then $\int_\gamma g_n = 0 \quad \forall\, n \geq 1$

$$\Rightarrow \lim_{n \to \infty} \int_\gamma g_n = \int_\gamma \lim g_n = \int_\gamma g = 0 \qquad \left(\because g_n \to g \text{ uniformly} \right)$$

Morera's thm $\Rightarrow$ g is holomp.

Let $z \in \Omega$. Choose $\epsilon > 0$ s.t $\overline{B(z,\epsilon)} \subseteq \Omega$, $\gamma = \partial B(z,\epsilon)$

$$g_n'(z) = \frac{1}{2\pi i}\int_\gamma \frac{g_n(w)}{w-z}\,dw$$

By uniform conv.,
$$\lim_{n\to\infty} \frac{1}{2\pi i}\int_\gamma \frac{g_n(w)}{w-z}\,dw$$
$$= \frac{1}{2\pi i}\int_\gamma \lim_{n\to\infty} \frac{g_n(w)}{w-z}\,dw$$
$$= \frac{1}{2\pi i}\int_\gamma \frac{g(w)}{w-z}\,dw = g'(z)$$

Lem 2: Let Ω be open & conn. Let $\{f_n\}_{n\geq 1}$ be a seq. of injective holomp. maps that conv. uniformly on every $K \subseteq \Omega$ compact to $f : \Omega \to \mathbb{C}$, then f is injective or const.

Pf: Suppose f is not inj. So, $\exists\, z_1, z_2$ $(z_1 \neq z_2)$ s.t $f(z_1) = f(z_2)$

Let $g_n(z) = f_n(z) - f_n(z_1)$

f_n is inj. $\Rightarrow$ g_n is inj.

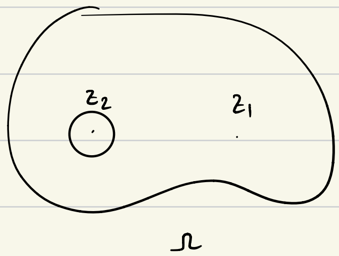
$g_n(z_1) = f_n(z_1) - f_n(z_1) = 0 \quad \Rightarrow \quad g_n(z) \neq 0 \quad \forall\, z \neq z_1$

Also, $g_n(z) \to f(z) - f(z_1)$

If f is <u>not</u> a const. f^{x^n}.

Then z_2 is also a zero of $f(z) - f(z_1)$ $\left(\because f(z_1) = f(z_2)\right)$

Def. $g(z) = f(z) - f(z_1)$



Note, $\displaystyle \frac{1}{2\pi i} \int_{\partial B(z_2, \epsilon)} \frac{g'(z)}{g(z)}\, dz \geq 1$ (by argument principle)

However, g_n has no zeros in $\overline{B(z, \epsilon)}$

$$\Rightarrow \frac{1}{2\pi i} \int_{\partial B(z_2, \epsilon)} \frac{g_n'(z)}{g_n(z)}\, dz = 0$$

By lem 1, $g_n' \to g'$ uniformly

$\dfrac{1}{g_n} \to \dfrac{1}{g}$ uniformly on $\partial B(z_2, \epsilon)$ because $g_n \neq 0$ & $g \neq 0$

$$\Rightarrow \frac{g_n'}{g_n} \to \frac{g'}{g} \text{ uniformly}$$

$$\Rightarrow \underbrace{\int \frac{g_n'}{g_n}}_{0} \to \underbrace{\int \frac{g'}{g}}_{\geq 1} \text{ uniformly} \to \text{Contd}^n$$

Lem 3 : Let Ω be open & simply conn. s.t $0 \notin \Omega$.
Then $\exists f : \Omega \to \mathbb{C}$ holomp. & $\exp(f(z)) = z \quad \forall z \in \Omega$
(terminology $f(z) = \log_\Omega(z)$)

Pf : Since $0 \notin \Omega$, $1/z$ is holomp. on Ω.
Fix $z_0 \in \Omega$. Def. $f(z) = \int_\gamma \frac{1}{w} \, dw + \text{Log}(z_0)$

where γ is a path from z_0 to z.

f is well-defined because $1/w$ is holomp. on Ω & Ω is simply conn.

Observe f is holomp. on Ω & $f'(z) = \frac{1}{z}$

Consider $g(z) = z \exp(-f(z))$
Then, $g'(z) = \exp(-f(z)) + z \exp(-f(z)) (-f'(z))$
$= 0$
$\Rightarrow g$ is const.

$$f(z_0) = \int_{\mathbb{1}_{z_0}} \frac{dw}{w} + \text{Log}(z_0) = \text{Log}(z_0)$$

$\Rightarrow \quad g(z) = g(z_0) = z_0 \exp(-f(z_0))$

We have $\exp(f(z_0)) = C \cdot z_0 \quad (\because g(z) \text{ is const.})$

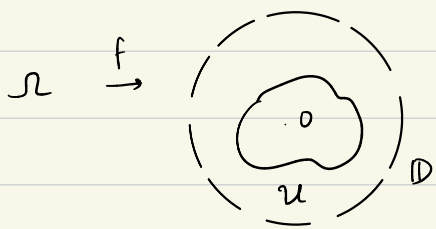
$\Rightarrow \exp(\operatorname{Log}(z_0)) = C \cdot z_0$

$\Rightarrow z_0 = C z_0 \Rightarrow C = 1$

$\Rightarrow z\exp(-f(z)) = 1 \quad \Rightarrow \exp(f(z)) = z$

Proof Structure of Riemann Mapping Thm

Step 1:



Find $f : \Omega \to \mathcal{U}$

biholomp   open simply conn. subset of $\mathbb{D}$ s.t. $0 \in \mathcal{U}$

Steps 2 & 3: 'Maximize' $\mathcal{U}$

Pf: Step 1: Since $\Omega \neq \mathbb{C}$, choose $\alpha \in \Omega \setminus \mathbb{C}$

Then $z - \alpha \neq 0 \quad \forall z \in \Omega$

Since Ω is simply conn., $\exists f$ on Ω holomp. & $\exp(f(z)) = z - \alpha$

Like earlier, we write $f(z) = \log_{\Omega}(z - \alpha)$

f(z) is inj.: Suppose $f(z) = f(w)$

$$\Rightarrow \exp(f(z)) = \exp(f(w))$$

$$\Rightarrow z - \alpha = w - \alpha$$

$$\Rightarrow z = w$$

Choose $w \in \Omega$.

Claim: $f(z) \neq f(w) + 2\pi i \quad \forall z \in \Omega$

Suppose $\exists z \in \Omega$ s.t $f(z) = f(w) + 2\pi i$

$$\Rightarrow \exp(f(z)) = \exp(f(w) + 2\pi i)$$

$$= \exp(f(w))$$

$$\Rightarrow z = w$$

$$\Rightarrow f(z) = f(w) \Rightarrow 0 = 2\pi i \rightarrow \text{Contd}^{n}$$

Infact, $\exists\ \epsilon>0$ s.t $B(f(w)+2\pi i,\epsilon)\cap f(\Omega)=\emptyset$

If not, $\exists\ \{z_n\}$ s.t $\lim_{n\to\infty} f(z_n) = f(w)+2\pi i$

$$\Rightarrow \exp\left(\lim_{n\to\infty} f(z_n)\right) = \exp(f(w)+2\pi i)$$

$$\Rightarrow \lim_{n\to\infty} \exp(f(z_n)) = \exp(f(w))$$

$$\Rightarrow \lim_{n\to\infty} z_n-\alpha = w-\alpha$$

$$\Rightarrow \lim_{n\to\infty} z_n = w$$

$$\Rightarrow f(\lim z_n) = f(w)$$

$$\Rightarrow \lim f(z_n) = f(w) \Rightarrow 0 = 2\pi i \rightarrow \text{Contd}^n$$

Hence, $|f(z)-(f(w)+2\pi i)| \geq \epsilon$

$$\Rightarrow \frac{1}{|f(z)-(f(w)+2\pi i)|} \leq \frac{1}{\epsilon} = B$$

<u>Note</u>: $$F(z) = \frac{1}{f(z)-(f(w)+2\pi i)}$$

f is inj. $\Rightarrow$ F is inj.

f is holomp. $\Rightarrow$ F is holomp. & $|F(z)|\leq B$

($\because$ denominator $\neq 0$)

Redefine f as $G(z) = \frac{1}{2B} f(z) - c$

s.t $G : \Omega \to G(\Omega)$ is biholomp. & $G(\Omega) \subseteq \mathbb{D}$ & $0 \in G(\Omega)$

What remains to show is that $\mathcal{U}$ can be taken to be $\mathbb{D}$.

At this pt., assume $\Omega \subseteq \mathbb{D}$.

$\mathcal{F} = \{ f : \Omega \to \mathbb{D}$ holomp., inj., $f(0) = 0 \}$

$f(z) = z$

Identity fxn denoted by $\mathbb{1} \in \mathcal{F}$

$\mathbb{1}'(0) = 1$

1. $\mathcal{F}$ is uniformly b'nd : $|f(z)| < 1 \ \forall z \in \Omega, f \in \mathcal{F}$

2. Also, $f'(0)$ is uniformly b'nd i.e $\exists c > 0$ s.t $|f'(0)| \leq c \ \forall f \in \mathcal{F}$

$$f'(0) = \frac{1}{2\pi i} \int_{\partial B(0,\epsilon)} \frac{f(z)}{z^2} \, dz$$

$$\Rightarrow |f'(0)| \leq \frac{1}{2\pi} \int_{\partial B(0,\epsilon)} \frac{|f(z)|}{\underbrace{|z|^2}_{\epsilon^2}} \, dz$$

$$\leq \frac{2\pi\epsilon}{2\pi\epsilon^2} \sup_{z\in\partial B(0,\epsilon)} |f(z)|$$

$$\leq \frac{\sup |f(z)|}{\epsilon} \xrightarrow{\text{may}} \infty \quad \text{as } \epsilon \to 0 \;!$$

But $f(0) = 0 \Rightarrow \dfrac{f(z)}{z}$ is holomp.

So, instead,

$$|f'(0)| \leq \frac{1}{2\pi} \int_{\partial B(0,\epsilon)} \frac{|f(z)/z|}{\underbrace{|z|}_{\epsilon}} \, dz$$

$$\leq \sup_{z\in\partial B(0,\epsilon)} \left| \frac{f(z)}{z} \right|$$

Construct $\{f_n\}_{n\geq 1} \subseteq \mathcal{F}$ s.t $\lim_{n\to\infty} |f_n'(z)| = S$

By Montel's thm, $\exists$ a subseq. of $\{f_n\}_{n\geq 1}$ which conv. to some $f(z)$ holomp., s.t conv. is uniform on every compact $K \subseteq \Omega$

1. Why is $f(0) = 0$?

because $\lim_{n\to\infty} f_n(0) = \lim_{n\to\infty} 0 = 0$

$\Rightarrow f(0) = 0$

2. Also, $|f_n(z)| < 1 \Rightarrow |f(z)| \leq 1 \quad \forall z \in \Omega$

Ω is open $\Rightarrow$ By max-mod. principle, $|f(z)| = 1$ for some $z \in \Omega$

means f is a const.

But $f(0) = 0 \Rightarrow f(\Omega) = 0 \Rightarrow f$ is not injective $\longrightarrow$ Contd^n

Why is f not const. ?

f is const. $\Rightarrow |f'(0)| = 0$, contradicting $S \geq 1$

$\Rightarrow f \in \mathcal{F}$

Claim: This is the required f i.e $f : \Omega \to \mathbb{D}$ is a biholomp.

If $f(\Omega) \neq \mathbb{D}$, then $\exists\, \alpha \in \mathbb{D} \setminus f(\Omega)$

Recall, $\psi_\alpha(z) = \dfrac{\alpha - z}{1 - \bar{\alpha} z}$, $\alpha \in \mathbb{D}$

$\psi_\alpha(0) = \alpha$, $\psi_\alpha(\alpha) = 0$, $\psi_\alpha \circ \psi_\alpha = 1$

Consider $\tilde{\Omega} = \psi_\alpha \circ f(\Omega)$ is simply conn.

Note, $0 \notin \tilde{\Omega}$ (because $0 \in \tilde{\Omega} \Rightarrow 0 = \psi_\alpha \circ f(z_0)$

$\Rightarrow \psi_\alpha(0) = f(z_0)$

$\Rightarrow \alpha = f(z_0) \to$ Contd^n)

$\tilde{\Omega}$ is simply conn.

$\Rightarrow \exists\, g(w) = \exp\left(\frac{1}{2} \log_{\tilde{\Omega}}(w)\right)$ well-defined, holomp. fx^n

Also, $g(w)^2 = \exp(\log_{\tilde{\Omega}}(w)) = w$

So, $\underbrace{h(z) = z^2}_{\text{not inj.}}$ is the inverse of $g(w)$.

Consider $F = \psi_{g(\alpha)} \circ g \circ \psi_\alpha \circ f$ inj., holomp.

& $F(0) = 0 \Rightarrow F \in \mathcal{F}$

Also, $f = \psi_\alpha \circ h \circ \psi_{g(\alpha)} \circ F = \Phi$

h is <u>not</u> inj. $\Rightarrow \Phi$ is <u>not</u> inj.

$\Rightarrow \Phi$ is <u>not</u> a rot^n

$\Rightarrow$ Schwartz lemma implies $|\Phi'(0)| < 1$

$f = \Phi \circ F$

$\Rightarrow f'(z) = \Phi'(F(z)) F'(z)$

$\Rightarrow |f'(0)| = |\Phi'(0)| \, |F'(0)|$

$\Rightarrow |f'(0)| < |F'(0)| \rightarrow$ Contd^n because then $|F'(0)| > s$

Theory of infinite products

Let $S = \{z_1, \ldots, z_n\}$

Orders : $k_1, \ldots, k_n$

We can construct fx^n having prescribed zeros with given orders.

What if S is ∞-set?

For $S = \mathbb{Z}$, $f(z) = \sin(\pi z) \to \prod_{k\in\mathbb{Z}} (z-k)$?

Def^n : An infinite prod. $\prod_{n=1}^{\infty} c_n$ is said to converge if
$\lim_{N\to\infty} \prod_{n=1}^{N} c_n$ exists.

Lem 1 : If $\{a_n\}_{n\geq 1} \subseteq \mathbb{C}$ s.t $\sum_{n=1}^{\infty} |a_n| < \infty$, then $\prod_{n=1}^{\infty} (1+a_n)$ exists,
and is 0 iff $(1+a_m) = 0$ for some $m \geq 1$.

Pf : $\sum |a_n| < \infty \Rightarrow \exists N \geq 1$ s.t $|a_n| < \frac{1}{2}$ $\forall n \geq N$.
$\Rightarrow \forall n \geq N$, $\log(1+|a_n|)$ has a power series expansion
& $\exp(\log(1+a_n)) = 1+a_n$

Consider $\prod_{n=1}^{M} (1+a_n) = \prod_{n=1}^{M} \exp(\log(1+a_n))$
$= \exp\left(\sum_{n=1}^{M} \log(1+a_n)\right)$

Remainder term from Taylor expansion

$$\Rightarrow \quad |\log(1+a_n)| \leq 2|a_n| \qquad (|a_n| < 1/2)$$

$$\sum_{n=1}^{\infty} |a_n| < \infty \Rightarrow \sum_{n=N}^{\infty} |\log(1+a_n)| < \infty \Rightarrow \sum_{n=1}^{\infty} |\log(1+a_n)| < \infty$$

$$\Rightarrow \lim_{N\to\infty} \exp\left(\sum_{n=1}^{N} \log(1+a_n)\right) = \exp\left(\lim_{N\to\infty} \sum_{n=1}^{N} \log(1+a_n)\right) \quad \text{is well-defined}$$

$$\Rightarrow \lim_{N\to\infty} \prod_{n=1}^{N} (1+a_n) \quad \text{exists}$$

Also, $\lim_{N\to\infty} \sum_{n=1}^{N} \log(1+a_n) = C$ & $\exp C \neq 0$

$$\Rightarrow \prod_{n=1}^{\infty} (1+a_n) = \underbrace{\prod_{n=1}^{N-1} (1+a_n)}_{\text{possibly } 0} \cdot \underbrace{\prod_{n=N}^{\infty} (1+a_n)}_{\text{never } 0}$$

$\sum_{n\geq 1} |a_n| < \infty$, then exists & is 0 iff $(1+a_m) = 0$ for some $m \geq 1$

Lem 2: Let $\Omega \underset{\text{open}}{\subseteq} \mathbb{C}$ & $\{f_n(z)\}_{n\geq 1}$, $f_n : \Omega \to \mathbb{C}$ s.t

$\exists \{c_n\}_{n\geq 1}$ with $c_n > 0$ & $|1 - f_n(z)| \leq c_n$ $\forall z \in \Omega$ & $\sum c_n < \infty$

Then

1. $\prod_{n=1}^{\infty} f_n(z)$ conv. uniformly on Ω to a holomp. fx^n $F(z)$.
2. $\dfrac{F'(z)}{F(z)} = \sum_{n=1}^{\infty} \dfrac{f_n'(z)}{f_n(z)}$ as long as none of $f_n(z)$ vanish

Pf: 1) Denote $a_n(z) = f_n(z) - 1$

$$\Rightarrow |a_n(z)| \leq c_n$$

$$\Rightarrow \prod_{n=1}^{\infty} \underbrace{(1 + a_n(z))}_{f_n(z)} \quad \text{conv. } \forall z \in \Omega$$

By Weierstrass M-test, conv. is uniform $\Rightarrow F(z)$ is also holomp.

2) Assume that none of $f_n(z)$ vanish

Denote $G_N(z) = \prod_{n=1}^{N} f_n(z)$ holomp.

$$G_N(z) \to F(z) \text{ uniformly } \forall \text{ compact } K \subseteq \Omega$$

By prev. lemma, $G_N'(z) \to F'(z)$ uniformly on every compact $K \subseteq \Omega$

Since K is compact, $F_n(z) \neq 0$ on K

$\Rightarrow \exists M_n > 0$ s.t. $|F_n(z)| \geq M_n \quad \forall z \in K$

$$\Rightarrow \quad \frac{1}{|F_n(z)|} \leq \frac{1}{M_n} = B_n \quad \forall z \in K$$

$$\Rightarrow \quad \frac{1}{G_N(z)} \longrightarrow \frac{1}{F(z)} \quad \text{uniformly}$$

$$\Rightarrow \quad \frac{G_N'(z)}{G_N(z)} \longrightarrow \frac{F'(z)}{F(z)} \quad \text{uniformly}$$

Weierstrass product formula

Given a seq. $\{a_n\}_{n\geq 1}$ s.t $|a_n| \to \infty$ as $n \to \infty$,

then $\exists$ an entire fx^n $f : \mathbb{C} \to \mathbb{C}$ which vanishes exactly

at a_ns with the prescribed multiplicity.

If f_1, f_2 are two such fx^{n}s, then $\exists$ entire $g : \mathbb{C} \to \mathbb{C}$ s.t

$f_1(z) = f_2(z)\, e^{g(z)}$

Pf : Let's first show that if f_1 & f_2 are 2 such fx^{n}s, then

$\exists\, g : \mathbb{C} \to \mathbb{C}$ entire s.t $f_1(z) = f_2(z)\, e^{g(z)}$

Consider $\dfrac{f_1(z)}{f_2(z)}$.

This vanishes $\underbrace{\text{nowhere}}_{\substack{\text{zeroes of } f_1 \\ \text{are exactly} \\ \text{cancelled by} \\ \text{zeroes of } f_2}}$ & is $\underbrace{\text{entire}}_{\substack{\because \text{ all singularities} \\ \text{are removable}}}$

Note, codomain $(\underbrace{f_1/f_2}_{f}) \subseteq \mathbb{C} \setminus \{0\}$

Then $\{f(z) : z \in \mathbb{C}\} \subseteq \mathbb{C} \setminus \{0\}$

open (by open mapping thm)

Obviously non-empty, so pick $z_0 \in \{f(z) : z \in \mathbb{C}\}$

Define $g(z) = \int_{\gamma_z} \frac{f'}{f}(w)\, dw + \log(z_0)$

when γ_z is a path from z_0 to z

The map is well defined as $\{f(z) : z \in \mathbb{C}\}$ is simply conn.

$\Rightarrow e^{g(z)} = \frac{f_1}{f_2}(z)$ & $g(z)$ is entire.

Construction of f_1

Not^n : Let a_ns be non-distinct & repeated exactly multiplicity many times.

Idea: Let $z_0 = 0$ have multiplicity m & none of the a_ns are 0.

Consider $z^m \prod_{n \geq 1} \underbrace{\left(1 - \frac{z}{a_n}\right)}_{E_n\left(\frac{z}{a_n}\right)}$ where $E_n(z)$'s are canonical factors.

$E_0(z) = (1-z)$

$\vdots$

$E_n(z) = (1-z) \exp\left(z + \frac{z^2}{2} + \frac{z^3}{3} + \dots + \frac{z^n}{n}\right)$

Note, $\log(1-z) = -\left(z+\frac{z^2}{2}+\cdots\right)$ for $|z|<1$

So, for $|z|<1$, $(1-z) = e^{\log(1-z)}$

$$= e^{-\left(z+\frac{z^2}{2}\cdots\right)}$$

Then, $E_n(z) = \exp\left(-\left(\frac{z^{n+1}}{n+1} + \frac{z^{n+2}}{n+2} + \cdots\right)\right)$

<u>Lem</u>: When $|z|\leq 1/2$, then $|1-E_n(z)| \leq \underset{\substack{\uparrow \\ \text{equals } 2e}}{C}|z|^{n+1}$ for some $C>0$

<u>Pf</u>: Since $|z|\leq 1/2 < 1$,

$$E_n(z) = \exp\left(-\left(\frac{z^{n+1}}{n+1} + \frac{z^{n+2}}{n+2} + \cdots\right)\right)$$

$$|z|\leq 1/2 \Rightarrow \left|-\left(\frac{z^{n+1}}{n+1} + \frac{z^{n+2}}{n+2} + \cdots\right)\right| \leq |z|^{n+1}\sum_{k=0}^{\infty}\frac{|z|^k}{n+1+k}$$

$$\leq |z|^{n+1}\sum_{k=0}^{\infty}\frac{1}{2^k} \leq 2|z|^{n+1}$$

$$\Rightarrow |1-E_n(z)| = \left|1 - e^{-\left(\frac{z^{n+1}}{n+1} + \frac{z^{n+2}}{n+2} + \cdots\right)}\right|$$

$$\leq e\left|-\left(\frac{z^{n+1}}{n+1} + \frac{z^{n+2}}{n+2} + \cdots\right)\right|$$

$$\leq 2e|z|^{n+1}$$

Point of calculation. Recall if $\sum |a_n| < \infty$, then $\prod(1-a_n) < \infty$

$a_n \leftrightarrow (1-a_n) \Rightarrow$ If $\sum |1-a_n| < \infty$, then $\prod a_n < \infty$

Note, since $|a_n| \to \infty$

fix $z \in \mathbb{C}$, divide $\{a_n\}$ into 2 parts.

$$\left|\frac{z}{a_n}\right| \leq \frac{1}{2} \quad \text{v/s} \quad \underbrace{\left|\frac{z}{a_n}\right| > \frac{1}{2}}_{\text{finite set}} \quad |a_n| < 2|z|$$

Then $$z^m \prod_{n \geq 1} E_n\left(\frac{z}{a_n}\right) = z^m \underbrace{\prod_{|a_n| < 2|z|} E_n\left(\frac{z}{a_n}\right)}_{\text{finite product}} \cdot \underbrace{\prod_{|a_n| \geq 2|z|} E_n\left(\frac{z}{a_n}\right)}_{\infty \text{ product}}$$

Since $\left|\frac{z}{a_n}\right| < \frac{1}{2}$ $\left|1 - E_n\left(\frac{z}{a_n}\right)\right| \leq \frac{2e}{2^{n+1}}$ & $\sum_{n \geq 1} \frac{2e}{2^{n+1}}$ conv.

$\Rightarrow \prod_{|a_n| \geq 2|z|} E_n\left(\frac{z}{a_n}\right)$ converges

$\Rightarrow f(z) = z^m \prod_{n \geq 1} E_n\left(\frac{z}{a_n}\right)$ is an entire fx^n.

$f(z) = 0$ exactly when $z = 0$ or $E_n\left(\frac{z}{a_n}\right) = 0$ at $z = a_n$ with multiplicity 1.

Thus, $f(z)$ is the required fx^n.

Picard's little thm : A non-const. entire fx^n takes values on $\mathbb{C}$, except at most one pt.

Def^n : Let $f : \mathbb{C} \to \mathbb{C}$ be entire & satisfies $|f(z)| \leq Ae^{B|z|^{\rho}}$ for some $A, B, \rho \geq 0$.

Then $\rho_f = \inf\{\rho : |f(z)| \leq Ae^{B|z|^{\rho}}\}$ is called the order of $f(z)$.