

**MA412 PROBLEM SET
SPRING 2026**

Week 0 (Preliminaries)

0.1. Prove the commutativity, associativity and distributive properties of $+$ and \cdot in \mathbb{C} .

0.2. Let $z_1, \dots, z_n \in \mathbb{C}$. The triangle inequality is

$$|z_1 + \dots + z_n| \leq |z_1| + \dots + |z_n|.$$

When do we have equality in the above? Justify your answer.

0.3. Prove the inequality $||z| - |w|| \leq |z - w|$ for $z, w \in \mathbb{C}$.

0.4. Prove the Cauchy–Schwartz inequality:

$$\left| \sum_{k=1}^n z_k w_k \right|^2 \leq \left(\sum_{k=1}^n |z_k|^2 \right) \left(\sum_{k=1}^n |w_k|^2 \right).$$

When does equality occur?

0.5. Show that if $f : \mathbb{C} \rightarrow \mathbb{C}$ is continuous at $z = a$, then so is $|f| : \mathbb{C} \rightarrow \mathbb{R}_{\geq 0}$, defined by $|f|(z) = |f(z)|$.

Week 1 and Week 2 (5 Jan – 16 Jan)

1.1. Let $a, b, c \in \mathbb{C}$. Prove that

$$|ab + bc + ac| \leq |a|^2 + |b|^2 + |c|^2.$$

Hint: Use Cauchy’s inequality.

1.2. Show that if $u : \mathbb{R}^2 \rightarrow \mathbb{R}$ has continuous partial derivatives of first order at $(x, y) \in \mathbb{R}^2$, then there exists a function $r_{x,y} : \mathbb{R}^2 \rightarrow \mathbb{R}$ satisfying

$$u(x + h, y + k) - u(x, y) = \frac{\partial u}{\partial x} h + \frac{\partial u}{\partial y} k + r_{x,y}(h, k),$$

and

$$\lim_{\sqrt{h^2+k^2} \rightarrow 0} \frac{r_{x,y}(h, k)}{\sqrt{h^2 + k^2}} = 0.$$

1.3. Is the function $f(z) = \bar{z}$ real differentiable? That is to say, is the function $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by $f(x, y) = (x, -y)$ differentiable?

1.4. Is the function $f(z) = \bar{z}$ complex differentiable? What about differentiability at a point $z = a$ for $a \in \mathbb{R}$?

- 2.1. We know that $i^2 = -1$. Compute all the values of $(-1)^i$ and i^{2i} . Are the set of values the same?
- 2.2. Compute the power series expansion of $f(z) = \frac{1}{(1-z)^2}$ centered at $z = 0$. What's the radius of convergence?
- 2.3. Verify the Cauchy–Riemann equations for $f(z) = z^2$.

Week 3 – 4 (19 Jan – 27 Jan)

- 3.1. Prove the chain rule: Let $f : \Omega_1 \rightarrow \mathbb{C}$ and $g : \Omega_2 \rightarrow \mathbb{C}$ be differentiable, such that $f(\Omega_1) \subset \Omega_2$. Then $g \circ f$ is differentiable on Ω_1 and

$$(g \circ f)'(z) = g'(f(z))f'(z).$$

- 3.2. Let $\Omega \subset \mathbb{R}^2$ be an open connected set and γ be a path in Ω . Let $p, q : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be two functions. Show that the line integral

$$\int_{\gamma} p dx + q dy,$$

depends only on the end-points of γ if and only if there exists a function $U(x, y)$ on Ω such that $\partial U/\partial x = p$ and $\partial U/\partial y = q$.

- 3.3. Show that if a function $f(z)$ is complex differentiable, then so is the function $g(z) = \overline{f(\bar{z})}$.
- 3.4. Show that the annulus $D(0, 1) \setminus D(0, 1/4)$ is not simply connected.
- 3.5. Is the function $u(x, y) = x^3 - 3xy$ harmonic? If so, find the harmonic conjugate $v(x, y)$ that satisfies $v(0, 0) = 0$.
- 3.6. Let $a \in \mathbb{C}$ be fixed. Find the radius of convergence of the series $\sum_{n \geq 0} a^{n^2} z^n$.

Week 5 – 6 (2 Feb – 13 Feb)

- 5.1. Compute $\int_{\gamma} \bar{z} dz$ where γ is the straight line path from $z = -i$ to $z = i$.
- 5.2. Compute $\int_{\gamma} \bar{z} dz$ where γ is the semi circle joining $z = -i$ to $z = i$ in the counterclockwise direction.
- 5.3. Compute $\int_{\gamma} z^{-1} dz$ where γ is the quarter circle joining $z = 1$ to $z = i$ in the counterclockwise direction.
- 5.4. Compute $\int_{\gamma} z^{-1} dz$ where γ is the rectangle $[1 - i, 1 + i, -1 + i, -1 - i, 1 - i]$.
- 5.5. Demonstrate that the function $u(x, y) = x^3 - xy^2$ cannot be the real part of a holomorphic function.

5.6. Find the conditions on a, b, c so that $u(x, y) = ax^2 + bxy + cy^2$ is the real part of a harmonic function.

5.7. Say $a, b \in \mathbb{R}$ be such that $a \neq 0$ or $b \neq 0$. Prove that $f(x, y) = (ax + by)^m$ is a harmonic function for $m = 1$, but not harmonic for $m \geq 2$.

6.1. Let f be a holomorphic function on the strip $-1 < y < 1$, $x \in \mathbb{R}$ with

$$|f(z)| \leq A(1 + |z|)^b, \quad A, b > 0 \text{ fixed real numbers,}$$

for all z in the strip. Show that for each $n \in \mathbb{Z}_{>0}$, there exists $A_n \geq 0$ so that

$$f^{(n)}(x) \leq A_n(1 + |x|)^b, \quad \text{for all } x \in \mathbb{R}.$$

Hint: Use Cauchy's inequality for bounding derivatives.

6.2. Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be an entire function satisfying $|f(z)| \leq e^{\operatorname{Re}(z)}$. Prove that $f(z) = ce^z$ for some constant c .

6.3. Weierstrass's theorem states that a continuous real valued function on $[0, 1]$ can be uniformly approximated by polynomials. Can every continuous complex valued function on the closed unit disc be approximated uniformly by polynomials in the variable z ?

6.4. Compute the power series expansion of $f(z) = \frac{1}{i+z}$ around $z = i$. What is the radius of convergence?

6.5. Use the fact that if $f(z)$ is a non-zero holomorphic function, then its zeros cannot have a limit point to show that $f(z) = \sin(1/z)$ is not holomorphic in any neighbourhood of $z = 0$.

6.6. Suppose $f(z)$ is holomorphic in the punctured disc $D(z_0, r) - \{z_0\}$ and satisfies $|f(z)| \leq A(z - z_0)^{-1+\epsilon}$ for some $\epsilon, A > 0$ and all $z \in D(z_0, r) - \{z_0\}$. Prove that z_0 is a removable singularity of $f(z)$.

6.7. Let $p(z)$ be a polynomial of degree n and $R > 0$ be such that $p(z)$ has no zeroes in the region $\{z : |z| \geq R\}$. Prove that

$$\frac{1}{2\pi i} \int_{\gamma} \frac{p'(z)}{p(z)} dz = n,$$

where γ is the path $\gamma(t) = Re^{it}$, $0 \leq t \leq 2\pi$.

Upto Week 9 (16 Feb – 06 Mar)

9.1. Compute the residue of $f(z) = \frac{\sin(z)}{z}$ at $z = 0$.

9.2. Compute the residue of $f(z) = \frac{\sin(z)}{z^2}$ at $z = 0$.

9.3. What is the nature of the isolated singularity of

$$f(z) = \frac{1}{e^z - 1} - \frac{1}{z},$$

at $z = 0$?

9.4. What are the singularities of the meromorphic function

$$f(z) = \frac{\pi^2}{\sin^2(\pi z)}?$$

9.5. Compute the principal part of

$$f(z) = \frac{\pi^2}{\sin^2(\pi z)},$$

at $z = n$ for a given $n \in \mathbb{Z}$.

WEEK 10 (10 MAR – 14 MAR)

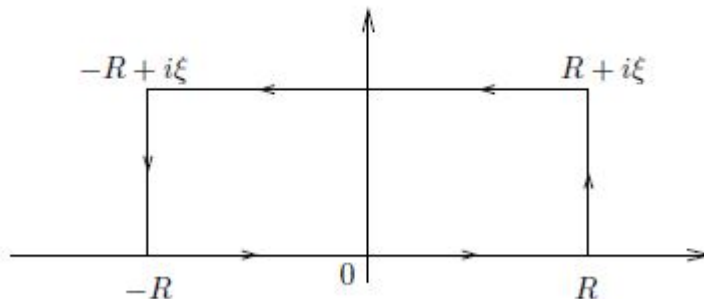
10.1. Prove that

$$\int_{-\infty}^{\infty} e^{-\pi x^2} dx = 1.$$

10.2. Prove that $f(x) = e^{-\pi x^2}$ is its own Fourier transform. That is, for $\xi \in \mathbb{R}$,

$$\int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i \xi x} dx = e^{-\pi \xi^2}.$$

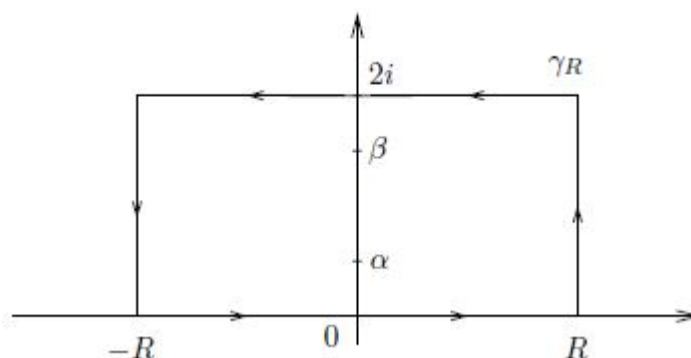
Use the following contour integral to prove this.



10.3. Let $\cosh(z) = \frac{e^z + e^{-z}}{2}$. Prove that $f(x) = \frac{1}{\cosh \pi x}$ is its own Fourier transform. That is, for $\xi \in \mathbb{R}$,

$$\int_{-\infty}^{\infty} \frac{1}{\cosh \pi x} e^{-2\pi i \xi x} dx = \frac{1}{\cosh \pi \xi}.$$

Use the following contour integral to prove this,



where $\alpha = i/2$ and $\beta = 3i/2$.

WEEK 11 – 12 (16 MAR – 27 MAR)

- 11.1. Use the Casorati–Weierstrass theorem to prove that all entire functions that are also injective take the form $f(z) = az + b$ with $a, b \in \mathbb{C}$ and $a \neq 0$.
- 11.2. Let $\Omega \subset \mathbb{C}$ be a connected set with non-empty interior (for simplicity, say Ω is connected and open), and let $H(\Omega)$ be the set of all holomorphic functions on Ω . Prove that $H(\Omega)$ is a ring. Moreover, it is an integral domain.
- 11.3. Let $\Omega \subset \mathbb{C}$ be an open set. Prove that if $H(\Omega)$ is an integral domain, then Ω must be connected.
- 11.4. Compute $\max_{z \in D(0,1)} |f(z)|$ for $f(z) = z^2 + 2z - 1$, where $D(0,1) = \{z : |z| \leq 1\}$. Where is the maxima achieved? Use the maximum modulus principle to argue that the maxima can occur at ∂D , and then parametrize ∂D to solve the problem.
- 12.1. Prove that for $|z| \leq 1/2$, $|\log(1+z)| \leq 2|z|$.
- 12.2. Find an example of a sequence of complex numbers $\{a_n\}$ such that $\sum a_n$ converges but $\prod(1+a_n)$ diverges. On the other hand, find a sequence $\{a_n\}$ such that $\prod(1+a_n)$ converges but $\sum a_n$ diverges.