MA5106

Fourier Analysis

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Grading policy : 2 Quizzes (15% each)
Midsem (25%)
Endsem (45%)

Curriculum : Fourier Series (Stein & Shakarchi)
Fourier Transform (R. Strichartz)

Let f be a (complex valued) fxn on $\mathbb{R}$ which is integrable & 2π-periodic.

$$f(\theta) = f(\theta + 2\pi) \quad \forall \ \theta \in \mathbb{R}$$

Def. $$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} \, dx$$

$\hat{f}(n)$ is called the <u>n^{th} Fourier coeff.</u> of f.

The (formal) series $\sum_{n=-\infty}^{\infty} \hat{f}(n) e^{inx}$ is called the <u>Fourier series</u> of f.

Consider the seq. of partial sums.

$$S_N(f) = \sum_{n=-N}^{N} \hat{f}(n) e^{inx}, \quad \forall \ N \in \mathbb{Z}_{\geq 0}$$

$$\{S_N(f)\}_{N \geq 1} \to g, \quad \text{then} \quad \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{inx} \to g$$

Fourier series are a part of a larger class called the trigonometric series which are expressions of the form $\sum_{n=-\infty}^{\infty} c_n e^{inx}$ for $c \in \mathbb{R}/\mathbb{C}$.

If a trig. series involves only finitely many terms, then it is called <u>trig. polynomial</u>.

The N^{th} partial sum of the Fourier series of f is the trig. poly.

$$S_N(f) = \sum_{n=-N}^{N} \hat{f}(n)\, e^{inx}, \qquad N \in \mathbb{Z}_{\geq 0}$$

In what sense does $S_N(f) \to f$ as $N \to \infty$?

eg: 1. $f(\theta) = \theta \quad \forall\ \theta = [-\pi, \pi)$ & $f(\pi) = -\pi$

$$\hat{f}(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)\, d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} \theta\, d\theta = 0$$

$$n \neq 0, \quad \hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \theta\, e^{-in\theta}\, d\theta = \frac{1}{2\pi} \left[\frac{-\theta e^{-in\theta}}{in} + \int \frac{e^{-in\theta}}{in}\, d\theta \right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}\left[\frac{-\theta e^{-in\theta}}{in} + \frac{e^{in\theta}}{n^2}\right]_{-\pi}^{\pi}$$

$$= \frac{(-1)}{2\pi}\left(\frac{\pi e^{-in\pi}}{in} - \frac{(-\pi)\, e^{-in(-\pi)}}{in}\right]$$

$$= \frac{(-1)}{2\pi}\left[\frac{\pi(-1)^n}{in} + \frac{\pi(-1)^n}{in}\right] \qquad (\because e^{i\pi} = -1)$$

$$= \frac{(-1)^{n+1}}{in}$$

Hence, the Fourier series is

$$\sum_{n\neq 0} \frac{(-1)^{n+1}}{in} e^{inx} = \sum_{k=1}^{\infty}\left[\frac{(-1)^{-k+1}\, e^{-ikx}}{i(-k)} + \frac{(-1)^{k+1}\, e^{ikx}}{ik}\right]$$

$$= 2\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \sin kx$$

Remarks:

1. $L^2[-\pi,\pi] := \left\{ f : [-\pi,\pi] \to \mathbb{C} : \int_{-\pi}^{\pi} |f|^2 dx < \infty \right\}$

$$\langle f, g\rangle := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,\overline{g(x)}\, dx$$

The set $\{e^{inx} : n \in \mathbb{Z}\}$ forms a countable orthonormal basis.

$\|f\|^2 = \langle f, f \rangle$ defines the L^2 norm

$$f \sim \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{inx} \iff \|f - S_N(f)\| \to 0 \text{ as } N \to \infty$$

Note that, $\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{inx}\,dx = 1$ for $n=0$

& $\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{inx}\,dx = 0$ for $n \neq 0$.

Hence, $$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{inx} \cdot \overline{e^{imx}}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(n-m)x}\,dx = \begin{cases} 1, & n=m \\ 0, & n \neq m \end{cases}$$

The space $L^2[-\pi, \pi]$ is separable.

2. If f is integrable, does the Fourier series of f conv. to f ptwise ?

This is <u>not</u> true because we can change fn^n f at one pt. w/o changing its Fourier coeffs.

3. There exist a cont. fn^n which is 2π-periodic, but its Fourier series diverges at a pt.

4. For cont. fn^{n}s, we can get uniform convergence for Cesaro & Abel sums of the Fourier series.

5. In 1966, L. Carleson showed that f is integrable & 2π-periodic, then the Fourier series of f converges to f at all pts. expect possibly for a set of measure zero.

Convolution

Given two 2π-periodic integrable fn's f & g on $\mathbb{R}$, we define their convolution $f * g$ on $[-\pi, \pi]$ by

$$(f * g)(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y)\, g(x-y)\, dy$$

Let f be a 2π-periodic int. fn. Then,

$$S_N(f) = \sum_{n=-N}^{N} \hat{f}(n) e^{inx} = \frac{1}{2\pi} \sum_{n=-N}^{N} \int_{-\pi}^{\pi} f(y) e^{-iny} dy\, e^{inx}$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) \left(\sum_{n=-N}^{N} e^{in(x-y)} \right) dy$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y)\, D_N(x-y)\, dy$$

$$= (f * D_N)(x)$$

where $D_N(x) = \sum_{n=-N}^{N} e^{inx}$ denotes the N^{th} Dirichlet kernel.

<u>Claim</u> : $D_N = \dfrac{\sin\left(N+\frac{1}{2}\right)x}{\sin x/2}$ if $x \neq 0$, $D_N(0) = 2N+1$

for $x \neq 0$,
$$D_N(x) = \frac{e^{i(-Nx)}\left(1-e^{ix(2N+1)}\right)}{1-e^{ix}}$$
$$= \frac{e^{-iNx} - e^{i(N+1)x}}{1-e^{ix}}$$
$$= \frac{e^{-i(N+\frac{1}{2})x} - e^{i(N+\frac{1}{2})x}}{e^{-ix/2} - e^{ix/2}}$$
$$= \frac{\sin\left(\frac{N+1}{2}\right)x}{\sin x/2}$$

<u>Good Kernels</u> :

A family of kernels $\{K_n(x)\}_{n=1}^{\infty}$ on the circle (i.e 2π-periodic) is said to be a family of <u>good kernels</u> if it satisfies the following properties:

1. $\forall\, n \geq 1$, $\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} K_n(x)\, dx = 1$

2. $\exists\, M > 0$ s.t $\forall\, n \geq 1$, $\displaystyle\int_{-\pi}^{\pi} |K_n(x)|\, dx \leq M$

3. $\forall \delta > 0$, $\displaystyle\int_{\delta \leq |x| \leq \pi} |K_n(x)|\, dx \to 0$ as $n \to \infty$

Thm:

Let f be a 2π-periodic int. fn & $\{K_n\}_{n=0}^{\infty}$ be a family of good kernels. Then,

$$\lim_{n\to\infty} (f * K_n)(x) = f(x)$$

whenever f is cont. at x.

If f is cont. everywhere, then the above lim. is uniform

Pf: Let $\epsilon > 0$. Let f be cont. at x

So, $\exists\, \delta > 0$ s.t $|y| < \delta \Rightarrow |f(x+y) - f(x)| < \epsilon$

$$(f * K_n)(x) - f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y) K_n(x-y)\, dy$$
$$- \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) K_n(y)\, dy \qquad \left(\because 1 = \frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(y)\, dy \right)$$

Note that $f * g = g * f$.

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(y)\,[f(x-y) - f(x)]\,dy$$

$$\Rightarrow |(f * K_n)(x) - f(x)| = \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(y)\,[f(x-y) - f(x)]\,dy\right|$$

$$\leq \frac{1}{2\pi}\int_{|y|<\delta} |K_n(y)|\,|f(x-y) - f(x)|\,dy$$

$$+ \frac{1}{2\pi}\int_{\delta\leq|y|\leq\pi} |K_n(y)|\,|f(x-y) - f(x)|\,dy$$

$\because$ f is integrable, $\exists\, D>0$ s.t $|f(x)| \leq D \;\; \forall\, x \in [-\pi, \pi]$

$$\therefore |(f * K_n)(x) - f(x)| \leq \frac{\epsilon}{2\pi}\int_{-\pi}^{\pi} |K_n(y)|\,dy + \frac{D}{\pi}\int_{\delta\leq|y|\leq\pi} |K_n(y)|\,dy$$

Since $\{K_n\}_{n=1}^{\infty}$ is a family of good kernels,

$\exists\, M>0$ s.t $\forall\, n \geq 0$, $\displaystyle\int_{-\pi}^{\pi} |K_n(y)|\,dy \leq M$ &

$\exists\, n_0 \in \mathbb{Z}_{>0}$ s.t $\forall\, n \geq n_0$, $\displaystyle\frac{D}{\pi}\int_{\delta\leq|y|\leq\pi} |K_n(y)|\,dy < \epsilon$

Hence, $|(f * K_n)(x) - f(x)| \leq C\epsilon \;\; \forall\, n \geq n_0$

i.e $(f * K_n)(x) \to f(x)$ as $n \to \infty$

This proves the first assertion of the thm.

If f is cont. on $[-\pi,\pi]$, then f is uniformly cont.
($\because$ $[-\pi,\pi]$ is compact)

In this case, δ can be chosen indep. of x, so
$f * K_n \to f$ uniformly ($\because$ n_0 is indep. of x)

$$D_N(x) = \sum_{n=-N}^{N} e^{inx}$$

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} D_N(x)\, dx = 1$$

However, $$\int_{-\pi}^{\pi} |D_N(x)|\, dx = \int_{-\pi}^{\pi} \frac{\left|\sin\left(N+\frac{1}{2}\right)x\right|}{|\sin(x/2)|}\, dx$$

($\because$ $|\sin(x)| \leq |x|$) $$\geq 2 \int_{-\pi}^{\pi} \frac{\left|\sin\left(N+\frac{1}{2}\right)x\right|}{|x|}\, dx$$

$\left(\text{Sub. } t = \frac{(2N+1)x}{2}\right)$ $$= 4 \int_{0}^{(2N+1)\frac{\pi}{2}} \frac{|\sin t|}{t}\, dt$$

$$\geq 4 \int_{0}^{N\pi} \frac{|\sin t|}{t}\, dt$$

$$\geq 4 \sum_{k=1}^{N} \int_{(k-1)\pi}^{k\pi} \frac{|\sin t|}{t} \, dt$$

$$\geq 4 \sum_{k=1}^{N} \int_{(k-1)\pi}^{k\pi} \frac{|\sin t|}{k\pi} \, dt$$

$$\left(\because \int_0^{\pi} \sin t \, dt = 2 \right) \qquad \geq \frac{8}{\pi} \sum_{k=1}^{N} \frac{1}{k}$$

As $N \to \infty$, RHS diverges

So, $\int_{-\pi}^{\pi} |D_N(x)| \, dx \to \infty$ as $N \to \infty$

Hence $\{D_n(x)\}_{n=0}^{\infty}$ is <u>not</u> a family of good kernels.

Since Fourier series does <u>not</u> conv. pt.wise at all pts., we consider a diff. sum.

Cesaro sum

Let $\sum_{k=0}^{\infty} c_n$ be a series of complex nos. & $S_n = \sum_{k=0}^{n} c_k$

The N^{th} Cesaro sum or mean of the series $\sum_{k=0}^{\infty} c_k$ is defined as $\sigma_n = \dfrac{s_0 + s_1 + \dots + s_{n-1}}{n}$

If σ_N conv. to a lim. σ as $N \to \infty$, then we say that $\sum_{n=0}^{\infty} c_n$ is Cesaro summable.

Rem: If a series conv., then it is also Cesaro summable

eg: Consider $\sum_{k=0}^{\infty} (-1)^k$. Then S_n's are $1, 0, 1, 0, \dots$
So, the series does not conv.
However, σ_n's are $1, \frac{1}{2}, \frac{2}{3}, \frac{1}{2}, \frac{3}{5}, \frac{1}{2}, \dots$
Thus, the Cesaro sum of $\sum_{k=0}^{\infty} (-1)^k$ is $\frac{1}{2}$.

The N^{th} Cesaro mean of the Fourier series of an int. fn^n f on the circle is

$$\sigma_N(f)(x) = \frac{S_0(f) + S_1(f) + \dots + S_{N-1}(f)}{N}$$

So, $$\sigma_N(f)(x) = \frac{1}{N}\sum_{k=1}^{N-1} S_k(f) = \frac{1}{N}\sum_{k=0}^{N-1} (f * D_k)(x)$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\, D_k(x-y)\, dy$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y) \left[\frac{1}{N}\sum_{k=0}^{N-1} D_k(x-y)\right] dy$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\, F_N(x-y)\, dy$$

$$= (f * F_N)(x)$$

where $F_N(x) = \frac{1}{N}\sum_{k=0}^{N-1} D_k(x)$ is the Fejer's kernel.

Lem: $F_N(x) = \dfrac{1}{N} \dfrac{\sin^2\left(\frac{Nx}{2}\right)}{\sin^2\left(\frac{x}{2}\right)}$ and $\{F_n\}_{n=0}^{\infty}$ is a family of good kernels.

Pf: Let $\omega = e^{ix}$.

Then
$$N F_N(x) = \sum_{k=0}^{N-1} \frac{e^{-kix} - e^{(k+1)ix}}{1-e^{ix}}$$

$$= \sum_{k=0}^{N-1} \frac{\omega^{-k} - \omega^{(k+1)}}{1-\omega}$$

$$= \frac{1}{1-\omega}\left(\frac{\omega^{-N+1}(1-\omega^N)}{1-\omega} - \frac{\omega(1-\omega^N)}{1-\omega} \right)$$

$$= \frac{1}{(1-\omega)^2}\left[(\omega^{-N+1} - \omega) - (\omega - \omega^{N+1})\right]$$

$$= \frac{1}{(1-\omega)^2}\left(\omega^{-N+1} - 2\omega + \omega^{N+1}\right) = \frac{\cancel{\omega}}{\cancel{\omega}} \frac{(\omega^{-N} - 2 + \omega^N)}{(\omega^{-1/2} - \omega^{1/2})^2}$$

$$= \left(\frac{\omega^{-N/2} - \omega^{N/2}}{\omega^{-1/2} - \omega^{1/2}}\right)^2 = \frac{\sin^2 Nx/2}{\sin^2 x/2}$$

Note: $\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} F_N(x)\, dx = 1$

For $\delta > 0$, $\delta \le |x| \le \pi$, $\exists\ c_\delta > 0$ s.t $\sin^2\left(\frac{x}{2}\right) > c_\delta$

Hence, $$NF_N(x) < \frac{1}{c_\delta} \Rightarrow F_N(x) < \frac{1}{Nc_\delta}$$

Moreover, $$\frac{1}{2\pi}\int_{\delta \le |x| \le \pi} F_N(x)\,dx < \frac{1}{Nc_\delta} \to 0 \quad \text{as } N \to \infty$$

Thus, $\{F_n\}_{n=0}^{\infty}$ satisfies the def^n of Good Kernel.

Thm: (Fejer's thm)

If f is an integrable fx^n on the circle, then the Fourier series of f is Cesaro summable to f at every pt. of cont. of f. Moreover, if f is (uniformly) cont., then the Fourier series is uniformly Cesaro summable to f.

Rem: 1. Cont. fx^{n}s on the circle can be uniformly approximated by trigonometric polynomials.

2. $$(f*g)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\,g(x-y)\,dy$$

Let $z = x-y$
$dz = -dy$

$$= -\frac{1}{2\pi}\int_{x+\pi}^{x-\pi} f(x-z)\,g(z)\,dz$$

$$= \frac{1}{2\pi}\int_{x-\pi}^{x+\pi} f(x-z)\,g(z)\,dz$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} g(z)\,f(x-z)\,dz$$

$$= (g*f)(x)$$

Abel mean

A series of complex nos. $\sum_{k=1}^{\infty} c_k$ is said to be Abel summable to s if for every $0 \le r < 1$, the series $A(r) = \sum_{k=0}^{\infty} c_k r^k$ conv. &

$$\lim_{r\to 1} A(r) = s$$

The quantity $A(r)$ is called the Abel mean of the series.

Rem: 1. If a series conv., then it is Abel summable.

2. If a series is Cesaro summable, then it is Abel summable.

eg: Consider $\sum_{k=0}^{\infty} (-1)^k (k+1) = 1-2+3-4+5\ldots$

Then $A(\lambda) = \sum_{k=0}^{\infty} (-1)^k (k+1) \lambda^k$

$$(1+\lambda)^{-2} = 1 + (-2)\lambda + \frac{(-2)(-3)}{2!}\lambda^2 + \frac{(-2)(-3)(-4)}{3!}\lambda^3 + \ldots$$

$$= 1 + (-1)(1+1)\lambda + (-1)^2(2+1)\lambda^2 + (-1)^3(3+1)\lambda^3 + \ldots$$

$$= A(\lambda)$$

$$\lim_{\lambda\to 1} A(\lambda) = \lim_{\lambda\to 1} \frac{1}{(1+\lambda)^2} = \frac{1}{2^2} = \frac{1}{4}$$

$$\{S_n\}_{n=1}^{\infty} = \{1, -1, 2, -2, 3, -3, \ldots\}$$

$$\{C_n\}_{n=1}^{\infty} = \{1, 0, \tfrac{2}{3}, 0, \tfrac{3}{5}, 0, \ldots\}$$

So, the series is <u>not</u> Cesaro summable

We define the Abel mean of an integrable fn^n f or its Fourier series $\sum_{n=-\infty}^{\infty} \hat{f}(n) e^{inx}$ by

$$A_r(f) = \sum_{n=-\infty}^{\infty} r^{|n|} \hat{f}(n) e^{inx} \quad (0 \le r < 1)$$

Note that, since f is integrable, $|\hat{f}(n)|$ is uniformly bounded.
i.e $\exists M > 0$ s.t $|\hat{f}(n)| < M \ \forall n \in \mathbb{Z}$

$$|\hat{f}(n)| = \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx \right| \le \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)| \, dx = M$$

$\left(\because f \text{ is int.}, \exists M \text{ s.t } |f(x)| < M \right)$

$$\Rightarrow |r^{|n|} \hat{f}(n) e^{inx}| \le M r^{|n|} \ \& \ \sum_{n=-\infty}^{\infty} M r^{|n|} < \infty$$

So, $A_r(f)$ conv. absolutely & uniformly for $0 \le r < 1$

Further, $$A_r(f)(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} \hat{f}(n) e^{in\theta} = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} r^{|n|} \int_{-\pi}^{\pi} f(x) e^{-inx} dx \, e^{in\theta}$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \left(\sum_{n=-\infty}^{\infty} r^{|n|} e^{in(\theta - x)} \right) dx$$

$$= (f * P_r)(\theta)$$

where P_r denotes the Poisson kernel given by

$$P_r(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta}$$

Lem: If $0 \leq r < 1$, then $P_r(\theta) = \dfrac{1-r^2}{1-2r\cos\theta + r^2}$

The Poisson kernel is a family of Good kernels as $r \to 1$ from below.

Pf: Let $\omega = r \cdot e^{i\theta}$. Then

$$P_r(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta} = \sum_{n=0}^{\infty} \omega^n + \sum_{n=1}^{\infty} \bar{\omega}^n = \frac{1}{1-\omega} + \frac{\bar{\omega}}{1-\bar{\omega}}$$

$$= \frac{1-\bar{\omega} + \bar{\omega} - |\omega|^2}{(1-\omega)(1-\bar{\omega})}$$

$$= \frac{1-r^2}{(1-re^{i\theta})(1-re^{-i\theta})} = \frac{1-r^2}{1-2r\cos(\theta)+r^2}$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta)\, d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta}\right) d\theta = 1$$

Also, $P_r(\theta) \geq 0$. Hence 1. & 2. of the def^n of Good kernel hold.

Rem: Note the modification in the def^n of Good kernels

So, for a fixed $\delta > 0$, if $\frac{1}{2} \leq r < 1$ & $\delta \leq |\theta| \leq \pi$, then

$\exists\ C_\delta > 0$ s.t $1 - \cos(\theta) > C_\delta$

$$\therefore\ 1 - 2r\cos(\theta) + r^2 \geq \underbrace{0}_{\min\limits_{\frac{1}{2} \leq r < 1} (1-r)^2} + \underbrace{(1 - \cos(\theta))}_{\min\limits_{\frac{1}{2} \leq r < 1} (2r(1-\cos(\theta)))} > C_\delta$$

So, $P_r(\theta) < \dfrac{1 - r^2}{C_\delta} \to 0$ as $r \to 1$

Hence, 3. of the def^n of Good kernel is satisfied.

This proves the following.

Thm: The Fourier series on an integrable fn^n on the circle is Abel summable to f at every pt. of continuity of f. Moreover, if f is cont. on the circle, then the Fourier series is uniformly Abel summable to f.

Cor: (of Fejer's thm)

If f is integrable on the circle & $\hat{f}(n)=0 \ \forall n \in \mathbb{Z}$,
then f is 0 at any pt. of continuity.

Steady state Heat eqⁿ

A twice continuously diff. real valued fnⁿ $u(x,y)$ is said to satisfy steady state heat eqⁿ if $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ or $\Delta u = 0$
where Δ is called the Laplace operator or Laplacian.

The solutions to the eqⁿ are called harmonic fnⁿs

Time dependent heat eqⁿ: $k\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$

$u(x,y)$ may denote the temperature at a pt. (x,y).

- Abel sum: $(f * P_r)(\theta) = \sum_{n=-\infty}^{\infty} \hat{f}(n) r^{|n|} e^{in\theta}$

Consider the unit disc in the plane

$$D = \{(x,y) \in \mathbb{R}^2 : x^2+y^2<1\}$$

whose b'ry is the unit circle C.

<u>The Dirichlet problem</u>: To solve the steady-state heat eq^n in the unit disc D subject to the b'ry cond^n $u=f$ on C where f is an integrable fx^n on the unit circle C.

<u>Thm</u>: Let f be an integrable fx^n on the unit circle. Then the fx^n u defined in the unit disc by the Poisson integral

$$u(r,\theta) = (f * P_r)(\theta)$$

has the following properties:

1. u is twice continuously diff. in the unit disc & $\Delta u = 0$.
2. If θ is any pt. of continuity of f, then

$$\lim_{r\to 1} u(r,\theta) = f(\theta)$$

If f is continuous everywhere, then the limit is uniform.

3. If f is continuous, then $u(r,\theta)$ is the unique sol^n to the steady state heat eq^n in the unit disc satisfying cond^n 2.

$$\left\{ \text{If } f \text{ is cont. on } C, \quad f = \lim_{n\to\infty} \sigma_n(f) \quad (\text{uniform conv.}) \right\}$$

Pf: $\quad u(r,\theta) = (f * P_r)(\theta) = \sum_{m=-\infty}^{\infty} a_m r^{|m|} e^{im\theta}$

where a_m's are the Fourier coeffs. of f.

For any fixed $\rho<1$, inside the disc of radius $r<\rho<1$ centered at origin, the series for u can be differentiated (wrt θ & r) term by term and the diff. series conv. uniformly & absolutely (in each case).

$$\left\{ \sum_{n=0}^{\infty} c_n x^n, \quad \lim_{n\to\infty} |c_n|^{1/n} = \lim_{n\to\infty} |nc_n|^{1/n} \right\}$$

Thus, u can be twice continuously differentiated (can be differentiated infinitely many times) inside the unit disc.

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x}$$

$r = \sqrt{x^2+y^2}$

$\theta = \tan^{-1}(y/x)$

$$= \frac{x}{\sqrt{x^2+y^2}}\frac{\partial u}{\partial r} - \frac{y}{x^2+y^2}\frac{\partial u}{\partial \theta}$$

$$= \cos(\theta)\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}$$

$$\frac{\partial u}{\partial y} = \sin(\theta)\frac{\partial u}{\partial r} + \frac{1}{r}\cos\theta\frac{\partial u}{\partial \theta}$$

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

i.e., $\Delta u = \sum\left(|m|(|m|-1)\, r^{|m|-2} e^{im\theta} + |m|\, r^{|m|-2} e^{im\theta} - |m|^2 r^{|m|-2} e^{im\theta}\right)$

$= 0$

This proves 1.

The proof of 2. follows from a prev. thm.

Suppose v solves the steady-state heat eqn on the disc & conv. to the given cont. fn f uniformly as $r \to 1$ from below.

For each fixed r with $0<r<1$, the fn $v(r,\theta)$ has a Fourier series $\sum_{n=-\infty}^{\infty} a_n(r) e^{in\theta}$ where $a_n(r) = \frac{1}{2\pi}\int_{-\pi}^{\pi} v(r,\theta)\, e^{-in\theta}\, d\theta$

Note that, $\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} = 0 \quad -(*)$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\partial^2 v}{\partial \theta^2} e^{-in\theta}\, d\theta = \underbrace{\frac{1}{2\pi}\frac{\partial v}{\partial \theta} e^{-in\theta}\Big|_{-\pi}^{\pi}}_{0} + \frac{in}{2\pi}\int_{-\pi}^{\pi} \frac{\partial v}{\partial \theta} e^{-in\theta}\, d\theta$$

$$= \underbrace{\frac{in}{2\pi} v e^{-in\theta}\Big|_{-\pi}^{\pi}}_{0} - \frac{n^2}{2\pi}\int_{-\pi}^{\pi} v e^{-in\theta}\, d\theta$$

$$= -n^2 a_n(r)$$

Further, $\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\partial^2 v}{\partial r^2} e^{-in\theta}\, d\theta + \frac{1}{2\pi}\frac{1}{r}\int_{-\pi}^{\pi} \frac{\partial v}{\partial r} e^{-in\theta}\, d\theta$

$$= \frac{1}{2\pi}\frac{d^2}{dr^2}\int_{-\pi}^{\pi} v e^{-in\theta}\, d\theta + \frac{1}{2\pi}\frac{1}{r}\frac{d}{dr}\int_{-\pi}^{\pi} v e^{-in\theta}\, d\theta$$

($\because$ v is twice continuously differentiable)

$$= a_n''(r) + \frac{1}{r} a_n'(r)$$

So, by $(*)$ we get $a_n''(r) + \frac{1}{r} a_n'(r) - \frac{n^2}{r^2} a_n(r) = 0$

Substituting r^m, we get $\quad m(m-1)+m-n^2=0$

$$\Rightarrow \quad m = \pm n \quad (\text{for } n \neq 0)$$

So, $a_n(r) = A_n r^{|n|} + B_n r^{-|n|}$ for $n \neq 0$.

For $n=0$, $\quad a_0(r) = A_0 + B_0 \log(r)$

Since v is bounded, $a_n(r)$s are b'nd. But r^{-n} & $\log(r)$ is not b'nd as $r \to 0$. So, B_ns are zero.

Since v conv. uniformly to f as $r \to 1$, we get

$$A_n = \lim_{r\to 1} a_n(r) = \lim_{r \to 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} v(r,\theta)\, e^{-in\theta}\, d\theta$$

$$= \int_{-\pi}^{\pi} \lim_{r\to 1} v(r,\theta)\, e^{-in\theta}\, d\theta$$

$$= \int_{-\pi}^{\pi} f(\theta)\, e^{-in\theta}\, d\theta = \hat{f}(n)$$

Hence, for each $0<r<1$, the Fourier series of u & v are same in any circle of radius r.

Since v is cont., by corollary of Fejer's thm, we conclude that $u=v$.

<u>Example 1</u>: $\ell^2(\mathbb{Z}) = \{ (\dots, a_{-n}, \dots, a_0, \dots, a_n, \dots) : a_i \in \mathbb{C}$ & $\sum_{n=-\infty}^{\infty} |a_n|^2 < \infty \}$

Add^n & scalar multip^n is defined component-wise.

For $A = (\dots, a_{-n}, \dots, a_0, \dots, a_n, \dots)$,

$$\|A\| = \left(\sum_{n=-\infty}^{\infty} |a_n|^2 \right)^{1/2}$$

The triangle inequality can be shown as follows:
Let $A, B \in \ell^2(\mathbb{Z})$. Def. $A_N = (\dots, 0, a_{-N}, \dots, a_0, \dots, a_N, 0, \dots)$ &
B_N similarly. Then $\|A_N + B_N\| \leq \|A_N\| + \|B_N\| \leq \|A\| + \|B\|$

By letting $N \to \infty$, we get $\|A+B\| \leq \|A\| + \|B\|$
Hence, $\ell^2(\mathbb{Z})$ is a vector sp.

Sim. to triangle ineq., we can show Cauchy-Schwarz

$$\left| \sum_{n=-\infty}^{\infty} a_n \bar{b}_n \right| \leq \|A\| \, \|B\|$$

Therefore, we can define the inner product

$$\langle A, B \rangle := \sum_{n=-\infty}^{\infty} a_n \bar{b}_n$$

<u>Note</u>: $\|A\| = \langle A, A \rangle^{1/2}$

Hilbert space: If a vector space with an inner product is complete wrt. norm induced by the inner product, then it is called a Hilbert space.

eg: $\ell^2(\mathbb{Z})$, $\mathbb{R}^n$, $\mathbb{C}^n$.

Example 2: Let R denote the set of all complex valued Riemann integrable fn's on $[0, 2\pi]$.

Def. $$\|f\| = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(\theta)|^2 \, d\theta \right)^{1/2}$$

One can prove Cauchy-Schwarz & triangle inequality for this example.

R is a vector sp. with pt. wise add^n & scalar multiplication.

$$(f+g)(\theta) = f(\theta) + g(\theta)$$
$$(\lambda f)(\theta) = \lambda \cdot f(\theta)$$

An inner product here is defined as

$$\langle f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) \cdot \bar{g}(\theta) \, d\theta$$

In order to ensure that $\|f\| = 0 \Rightarrow f = 0$, we assume $f \sim g$ if $f - g$ vanishes except on a set of 'measure zero'.

- A set $A \subseteq \mathbb{R}$ is said to be of 'measure zero' if
$$\inf \left\{ \sum_{n=1}^{\infty} (b_n - a_n) : A \subset \bigcup_{n=1}^{\infty} [a_n, b_n] \right\} = 0$$

eg: singleton sets, $\mathbb{Z}$, $\mathbb{Q}$ are of measure zero.

Note that R is <u>not</u> complete.

Thm: Let f be an int. fxn on the circle.

Then $\|f - S_N(f)\| \to 0$ as $N \to \infty$ i.e $\frac{1}{2\pi}\int_0^{2\pi} |(f - S_N(f))(\theta)|^2 \, d\theta \to 0$

Pf: Set $e_n = e^{in\theta}$

Clearly, $\langle e_n, e_m \rangle = \begin{cases} 1, & \text{if } n = m \\ 0, & \text{otherwise} \end{cases}$

Let $a_n = \hat{f}(n)$ for $n \in \mathbb{Z}$.

$$\langle f, e_n \rangle = \frac{1}{2\pi}\int_0^{2\pi} f(\theta) e^{-in\theta} \, d\theta = \hat{f}(n) = a_n$$

$$\langle f - S_N(f), e_m \rangle = \Big\langle f - \sum_{|n| \leq N} a_n e_n, e_m \Big\rangle \quad \text{for } |m| \leq N$$
$$= \langle f, e_m \rangle - a_m = 0$$

i.e, $f - S_N(f) \perp e_m \quad \forall \ |m| \leq N$

i.e, $f - S_N(f) \perp \sum_{|n| \leq N} b_n e_n$ for any $b_n \in \mathbb{C}$. $-(*)$

Lem: (Best approximation)

If f is integrable on the circle with Fourier coeff. a_n, then $\|f - S_N\| \leq \|f - \sum_{|n|\leq N} c_n e_n\|$

Pf: $\|f - \sum_{|n|\leq N} c_n e_n\|^2 = \|f - \sum_{|n|\leq N} a_n e_n + \sum_{|n|\leq N} (a_n - c_n) e_n\|^2$

$$= \|f - \sum_{|n|\leq N} a_n e_n\|^2 + \|\sum_{|n|\leq N} (a_n - c_n) e_n\|^2 \quad (\text{using } *)$$

$$= \|f - \sum_{|n|\leq N} a_n e_n\|^2 + \sum_{|n|\leq N} (a_n - c_n)^2$$

$$\geq \|f - S_N(f)\|^2$$

Hence, the lemma holds.

Suppose f is cont. on the circle. Then by Fejer's thm, given $\epsilon > 0$, $\exists$ a trigonometric polynomial $P(\theta)$ say of degree m s.t $|f(\theta) - P(\theta)| < \epsilon \;\; \forall \theta \in [0, 2\pi]$

$$\Rightarrow \quad \|f - P\| \leq \left(\frac{1}{2\pi}\int_0^{2\pi} |f(\theta) - P(\theta)|^2 \, d\theta\right)^{1/2} < \epsilon$$

Hence by prev. lemma, $\|f - S_n(f)\| < \epsilon \;\; \forall \; n \geq N$

This proves the Thm when f is cont.

Lem: (Appendix lemma 1.5 Stein & Shakarchi)

Suppose f is int. on the circle & is bounded by B.

Then $\exists$ a seq. $\{f_k\}_{k=1}^{\infty}$ of cont. fn's on the circle s.t

$\sup_{x\in[0,2\pi]} |f_k(x)| \leq B \ \forall k \in \mathbb{Z}_{>0}$ & $\int_0^{2\pi} |f(x)-f_k(x)|\,dx \to 0$ as $k \to \infty$.

(Pf of Thm continued)

Let f be merely integrable.

Let $B = \sup_{\theta\in[0,2\pi]} |f(\theta)|$ & let $\epsilon > 0$. By prev. lemma, $\exists$ continuous fn g on the circle s.t

$$\sup_{\theta\in[0,2\pi]} |g(\theta)| < B \quad \& \quad \int_0^{2\pi} |f(\theta)-g(\theta)|\,d\theta \leq \frac{\epsilon^2\pi}{4B}$$

Now, $$\|f-g\|^2 = \frac{1}{2\pi}\int_0^{2\pi} |f(\theta)-g(\theta)|^2\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} |f(\theta)-g(\theta)|\,|f(\theta)-g(\theta)|\,d\theta$$

$$\leq \frac{2B}{2\pi}\underbrace{\int_0^{2\pi} |f(\theta)-g(\theta)|\,d\theta}_{\leq \frac{\epsilon^2\pi}{4B}}$$

$$\leq \frac{\epsilon^2}{4}$$

$\Rightarrow \|f-g\| \leq \epsilon/2$

Since g is cont., $\exists$ a trigonometric poly. P of deg m s.t

$\|g-P\|<\epsilon/2$

So, $\|f-P\| \leq \|f-g\|+\|g-P\| < \epsilon$

Hence, by Best Approx. lemma, $\|f-S_N(f)\|<\epsilon \quad \forall\ N \geq m$

This completes the proof.

Recall: $\ell^2(\mathbb{Z})$ & R

We have
$$\|f\|^2 = \|f-S_N(f)\|^2 + \|S_N(f)\|^2$$
$$= \|f-S_N(f)\|^2 + \sum_{|n|\leq N} |a_n|^2$$

Since $\|f-S_N(f)\| \to 0$ as $N\to\infty$, we get

$$\|f\|^2 = \sum_{n=-\infty}^{\infty} |a_n|^2$$

This is Parseval's identity

An orthonormal set of a complete inner product space (i.e Hilbert space) which satisfies Parseval's identity is called an orthonormal basis of the Hilbert space.

The set $\{ (\ldots, 0, 1, 0, \ldots) : n \in \mathbb{Z} \}$ is an onb of $\ell^2(\mathbb{Z})$.
↑ nth position

<u>Rem</u>: $\ell^2(\mathbb{Z})$ is complete but R is <u>not</u> complete.
So, $\exists \{a_n\}_{n=-\infty}^{\infty} \in \ell^2(\mathbb{Z})$ s.t $\nexists f \in R$ whose Fourier coeffs. are a_ns.

<u>Thm</u>: (Riemann-Lebesgue Lemma)
If f is integrable on the circle, then $\hat{f}(n) \to 0$ as $|n| \to \infty$

<u>Pf</u>: Since $\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 < \infty$, it follows that $|\hat{f}(n)| \to 0$ as $n \to \infty$
i.e $\hat{f}(n) \to 0$ as $n \to \infty$

Pp^n: Suppose f, g & h are integrable fn^{n}s on the circle.

Then

1. $f*(g+h) = (f*g)+(f*h)$
2. $(cf)*g = c(f*g) = f*(cg)$
3. $f*g$ is cts.

Pf: 1. & 2. Trivial

3. Assume g is cts.

Let $\epsilon > 0$ be given. Since g is cts., it is uniformly cts. on $[-\pi, \pi]$.

So, $\exists\, \delta > 0$ s.t $|s-t| < \delta \Rightarrow |g(s)-g(t)| < \epsilon$

Let $x_1, x_2 \in [-\pi, \pi]$ & $|x_1 - x_2| < \delta$.

$$|(f*g)(x_2) - (f*g)(x_2)| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(y)| \cdot |g(x_1-y) - g(x_2-y)|\, dy$$

$$< \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(y)| \cdot \epsilon\, dy = \frac{\epsilon}{2\pi}\int_{-\pi}^{\pi} |f(y)|\, dy$$

Since f is intgrable, we get $f*g$ is cts. in this case.

Next, assume that g is integrable (not necessarily cts.).

Apply a prev. lemma to g to obtain cts. fn's $\{g_k\}_{k=1}^{\infty}$ of approximating fn's.

Then, $$|(f*g - f*g_k)(x)| = |f*(g-g_k)(x)|$$
$$\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(y)|\cdot|g(x-y)-g_k(x-y)|\,dy$$
$$\leq \frac{1}{2\pi}\sup_{y\in[-\pi,\pi]}|f(y)|\int_{-\pi}^{\pi}|g(x-y)-g_k(x-y)|\,dy$$
$$= \frac{1}{2\pi}\sup_{y\in[-\pi,\pi]}|f(y)|\int_{-\pi}^{\pi}|g(z)-g_k(z)|\,dz$$
$$\to 0 \quad \text{uniformly as } k\to\infty$$

Since $f*g_k$ are cts., it follows that $f*g$ is cts.

$\underline{\text{Pp}^n}$: Suppose f, g & h are integrable fn^{n}s on the circle.
Then

1. $\widehat{f*g}(n) = \hat{f}(n) \cdot \hat{g}(n) \quad \forall n \in \mathbb{Z}$
2. $(f*g)*h = f*(g*h)$

$\underline{\text{Pf}}$: Assume f, g & h are cts.

$$\widehat{f*g}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} (f*g)(x)\, e^{-inx}\, dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\cdot g(x-y)\, dy\, e^{-inx}\, dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\frac{e^{-iny}}{2\pi}\int_{-\pi}^{\pi} g(x-y)\, e^{-in(x-y)}\, dx\, dy \qquad \left(\text{Fubini's Thm}\right)$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\frac{e^{-iny}}{2\pi}\int_{-\pi}^{\pi} g(z)\, e^{-inz}\, dz\, dy$$

$$= \hat{f}(n)\cdot\hat{g}(n)$$

$$((f*g)*h)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} (f*g)(z)\, h(x-z)\, dz$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\, g(z-y)\, dy\, h(x-z)\, dz$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{f(y)}{2\pi}\int_{-\pi}^{\pi} g(z-y)\, h(x-z)\, dz\, dy$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{f(y)}{2\pi}\int_{-\pi}^{\pi} g(\underbrace{z-y}_{w})\, h((x-y)-(\underbrace{z-y}_{w}))\, dz\, dy$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)(g*h)(x-y)\, dy$$

$$= f*(g*h)(x)$$

<u>Pf</u>: For the general case where f, g, h are <u>not</u> necessarily cts.

$\exists \{f_k\}_{k=1}^{\infty}, \{g_k\}_{k=1}^{\infty}, \{h_k\}_{k=1}^{\infty}$ be the seq. of cts. fn^{n}s on the circle approximating f, g, h resp. by lemma proved earlier.

$$\begin{aligned} |\widehat{f*g}(n) - \hat{f}(n)\cdot\hat{g}(n)| &\le |\widehat{f*g}(n) - \widehat{f_k*g_k}(n)| \\ &\quad + |\widehat{f_k*g_k}(n) - \hat{f}(n)\cdot\hat{g}(n)| \\ &= |\widehat{f*g}(n) - \widehat{f_k*g_k}(n)| \\ \text{(by prev. part)} &\quad + |\hat{f}_k\cdot\hat{g}_k(n) - \hat{f}(n)\cdot\hat{g}(n)| \end{aligned}$$

$$\begin{aligned} f*g - f_k*g_k &= (f*g - f_k*g) + (f_k*g - f_k*g_k) \\ &= (f-f_k)*g + f_k*(g-g_k) \end{aligned}$$

by the argument similar to the pp^n ($f*g$ is cont.),

$f_k*g_k \to f*g$ uniformly

Let $\epsilon > 0$, $\exists k_0$ s.t $\forall k \geq k_0$, $\sup |(f_k * g_k)(x) - (f*g)(x)| < \epsilon/2$

$$(f_k * g_k - f_k * g)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f_k(y)|\,|(g_k - g)(x-y)|\,dy$$

$$(\exists B \text{ s.t } |f_k(y)| \leq B) \qquad \leq \frac{B}{2\pi}\int_{-\pi}^{\pi} |g_k(z) - g(z)|\,dz$$

$$< \epsilon/2 \qquad (\exists k_1 \text{ s.t } \forall k \geq k_1)$$

For $k \geq \max\{k_0, k_1\}$,

$$|\widehat{f*g}(n) - \widehat{f_k * g_k}(n)| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |(f*g(x) - f_k * g_k(x))\, e^{-inx}|\,dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} |(f*g)(x) - (f_k * g_k)(x)|\,dx$$

Since $f_k * g_k \to f*g$ uniformly, we get $\widehat{f*g}(n) \to \widehat{f_k * g_k}(n)$

$$|\hat{f}(n) - \hat{f_k}(n)| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x) - f_k(x)|\,dx \to 0$$

$\Rightarrow \hat{f_k}(n) \to \hat{f}(n)$

Similarly, $\hat{g_k}(n) \to \hat{g}(n)$

Thus $\hat{f_k}(n) \cdot \hat{g_k}(n) \to \hat{f}(n) \cdot \hat{g}(n)$

For associativity,

$$(f*g)*h - (f_k*g_k)*h_k = [(f*g)*h - (f_k*g_k)*h] + [(f_k*g_k)*h - (f_k*g_k)*h_k]$$

Both of these can be made $< \epsilon/2$, hence

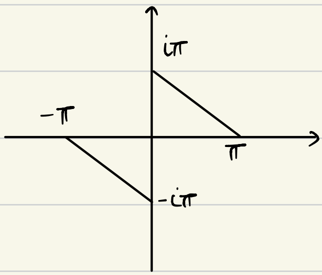
$$(f_k*g_k)*h_k \to (f*g)*h$$

Sim. $f_k*(g_k*h_k) \to f*(g*h)$.

Combining, we get the result.

Sawtooth fn^n

$$f(\theta) = \begin{cases} i(\pi - \theta) & \text{for } 0 < \theta \leq \pi \\ i(-\pi - \theta) & \text{for } -\pi \leq \theta < 0 \end{cases}$$



& $f(0) = 0$

The Fourier series of f is $\displaystyle\sum_{n \neq 0} \frac{e^{in\theta}}{n}$

$$S_N(f)(\theta) = f_N(\theta) = \sum_{0 < |n| \leq N} \frac{e^{in\theta}}{n}$$

$$\tilde{f}_N(\theta) = \sum_{n=-N}^{-1} \frac{e^{in\theta}}{n}$$

Objective: To construct a cts. fn^n on the circle whose fourier series at 0 diverges.

Claim: $\sum_{n=-\infty}^{-1} \frac{e^{in\theta}}{n}$ is not a Fourier series of a Riemann int'ble fx^n

Pf: Assume that it is the Fourier series of a Riemann int'ble fx^n $\tilde{f}$.

Then $A_r(\tilde{f})(\theta) = \sum_{n=-\infty}^{\infty} \frac{r^{|n|} e^{in\theta}}{n}$

$\therefore$ $|A_r(\tilde{f})(\theta)| = \sum_{n=1}^{\infty} \frac{r^n}{n} \to \infty$ as $r \to 1$

$\left(\because \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges} \right)$

But $|A_r(\tilde{f})(0)| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |\tilde{f}(\theta)| \, P_r(0-\theta) \, d\theta$

$\leq \sup |\tilde{f}(\theta)|$

This is a contd^n.

Claim: 1. $|\tilde{f}_N(0)| \geq \log N$

2. $f_N(\theta)$ is uniformly b'nd wrt N & θ.

$$\left(\tilde{f}_N(\theta) = \sum_{n=-N}^{-1} \frac{e^{in\theta}}{n} , \quad f_N(\theta) = \sum_{0<|n|\leq N} \frac{e^{in\theta}}{n} \right)$$

Pf: For 1., note that $|\tilde{f}_N(0)| = \sum_{n=1}^{N} \frac{1}{n} \geq \int_1^{N+1} \frac{dx}{x} \geq \int_1^{N} \frac{dx}{x}$

$$\geq \log(N)$$

For 2, we need the following lemma.

Lem: Suppose that the Abel means $A_r = \sum_{n=1}^{\infty} r^n c_n$ of the series $\sum_{n=1}^{\infty} c_n$ are bounded as $r \to 1$ from below.

If $c_n = O(1/n)$, then the partial sums $S_N = \sum_{n=1}^{N} c_n$ are bounded.

Pf: $r = 1 - 1/N$. Since $c_n = O(1/n)$, $\exists\, n_0 \in \mathbb{Z}_{>0}$ & $M > 0$ s.t

$$|c_n| \leq \frac{M}{n} \quad \forall\, n \geq n_0$$

For $N \geq n_0$, $|S_N - A_r| = \left| \sum_{n=1}^{N} (c_n - r^n c_n) + \sum_{n=N+1}^{\infty} r^n c_n \right|$

$$(1-r^n) = (1+r+\dots+r^{n-1})(1-r)$$
$$\leq n(1-r) \qquad [\because r<1]$$

$$\Rightarrow |S_N - A_r| \leq \sum_{n=1}^{N} n(1-r)|c_n| + \frac{M}{N}\sum_{n=N+1}^{\infty} r^n$$
$$\leq \sum_{n=1}^{N} M(1-r) + \frac{M}{N}\sum_{n=0}^{\infty} r^n$$
$$= MN(1-r) + \frac{M}{N(1-r)}$$
$$= 2M \qquad \left[\because r = 1-\frac{1}{N}\right]$$

Since A_r's are also bounded as $r \to 1$ from below, we conclude that S_N's are bounded.

We apply the lemma to the series $\sum_{n\neq 0} \frac{e^{in\theta}}{n}$ with
$$c_n = \frac{e^{in\theta}}{n} + \frac{e^{-in\theta}}{(-n)} \quad \forall\, n \in \mathbb{Z}_{>0}$$

$$\Rightarrow |c_n| \leq \frac{2}{|n|} \quad \text{i.e.} \quad c_n = O\left(\frac{1}{|n|}\right)$$

$$|A_r(f)(\theta)| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(y)|\, P_r(\theta-y)\, dy \leq \sup|f(y)| \cdot \frac{\cancel{2\pi}}{\cancel{2\pi}}$$

So, $f_N(\theta) = S_N(f)(\theta)$ is uniformly b'nd wrt N & θ by prev. lemma.

Next, define $P_N(\theta) = e^{i(2N)\theta} f_N(\theta)$, $\tilde{P}_N(\theta) = e^{i(2N)\theta} \tilde{f}_N(\theta)$

(degree $3N$) (degree $2N-1$)

<u>Lem</u> : $S_M(P_N) = \begin{cases} P_N, & M \geq 3N \\ \tilde{P}_N & M = 2N \\ 0 & M < N \end{cases}$

<u>Pf</u> : Clear because $P_N(\theta) = \sum_{n=-N}^{N} \frac{e^{i(n+2N)\theta}}{n}$

Claim: If $\alpha_k = 1/k^2$ & $N_k = 3^{2^k}$ for $k \in \mathbb{Z}_{>0}$, then the fxn
$g(\theta) = \sum_{k=1}^{\infty} \alpha_k P_k(\theta)$ is the desired fxn.

Pf: We have $|P_N(\theta)| = |e^{i(2N)\theta} f_N(\theta)| = |f_N(\theta)|$

Since f_N's are uniformly b'nd, so is P_N i.e $\exists B > 0$ s.t $|P_N(\theta)| < B$

$\therefore$ $|\alpha_k P_{N_k}(\theta)| \leq \frac{B}{k^2}$

So, $\sum_{k=1}^{\infty} \alpha_k P_{N_k}(\theta)$ conv. uniformly by Weierstrass M-test.

Since $\sum_{k=1}^{n} \alpha_k P_{N_k}(\theta)$ are cts. & periodic, the fxn g is cts.
& periodic.

Since $3^{2^{k+1}} > 3^{2^k+1} = 3 \cdot 3^{2^k}$, we have $N_{k+1} > 3N_k$

<u>Claim</u>: $|S_{2N_m}(g)(0)| > \alpha_m \log N_m + C$

<u>Pf</u>: $|S_{2N_m}(\alpha_m P_{N_m})(0)| = |\alpha_m \tilde{P}_{N_m}(0)| > \alpha_m \log N_m$

$(|\tilde{f}_N(0)| > \log N)$

prev. lemma

for $k < m$, $2N_m > 2 \cdot 3 N_k$

$$\therefore \left| S_{2N_m}\left(\sum_{k=1}^{m-1} \alpha_k P_{N_k}(\theta) \right) \right| = \left| \sum_{k=1}^{m-1} \alpha_k P_{N_k}(f)(\theta) \right|$$
$$\leq \sum_{k=1}^{m-1} \frac{B}{k^2} \leq \frac{\pi^2 B}{6}$$

For $k' > m$, $2N_m < \frac{2N_{k'}}{3} < N_{k'}$

$$\left| S_{2N_m}\left(\alpha_{k'} P_{N_{k'}}(\theta) \right) \right| = 0 \qquad \text{(by prev. lemma)}$$

So, the claim holds.

Since $\alpha_k \log N_k = \frac{1}{k^2} \log 3^{2^k} = \frac{2^k}{k^2} \log(3) \to \infty$ as $k \to \infty$

Hence, $|S_{2N_m}(g)(0)| \to \infty$ as $m \to \infty$

Rem: 1. If f is diff. at a pt. θ_0, then

$$S_N(f)(\theta_0) \to f(\theta_0) \text{ as } N \to \infty$$

2. If f is ctsly diff. on the circle, then the Fourier series of f is absolutely & uniformly convergent.

(Rectified)

Lem: Given $\sum_{n=1}^{\infty} c_n$ s.t $A_n = \sum_{n=1}^{\infty} r^n c_n$ is b'nd as $r \to 1$ from below. Also, $c_n = O(\frac{1}{n})$.

Then, for $S_N = \sum_{n=1}^{N} c_n$, $\{S_N\}_{N=1}^{\infty}$ is a b'nd seq.

Pf: Let $r = 1 - \frac{1}{N}$.

$\exists\, M_1 > 0$ & $n_0 \in \mathbb{Z}_{>0}$ s.t $|c_n| < \frac{M_1}{n}$ $\forall\, n \geq n_0$

Let $M = \max\{|c_1|, 2|c_2|, \dots, n_0|c_{n_0}|, M_1\}$

$\forall\, N \geq n_0$,

$$|S_n - A_r| = \left| \sum_{n=1}^{N} c_n - \sum_{n=1}^{\infty} r^n c_n \right|$$

$$= \left| \sum_{n=1}^{N} c_n(1-r^n) + \sum_{n=N+1}^{\infty} r^n c_n \right|$$

$$\leq \sum_{n=1}^{N} (1-r^n)|c_n| + \sum_{n=N+1}^{\infty} r^n |c_n|$$

$$\leq \sum_{n=1}^{N} n(1-r)|c_n| + \sum_{n=N+1}^{\infty} r^n |c_n| \qquad (\because (1-r^n) = (1-r)(1+r+\dots+r^{n-1}) \leq n(1-r))$$

$$\leq \sum_{n=1}^{N} (1-r)M + \sum_{n=N+1}^{\infty} r^n |c_n|$$

Distribution theory

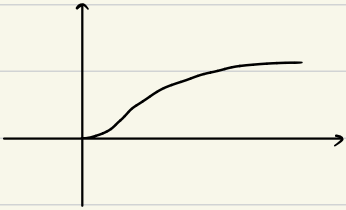
Let Ω be an open set in $\mathbb{R}^n$ & let $f: \Omega \to \mathbb{R}^n$.
Then the support of f, denoted by supp$f$, is defined as the closure of $\{x \in \Omega : f(x) \neq 0\}$

Defn: Let $\Omega \subseteq \mathbb{R}^n$ be an open set. The class of test fn's $\mathcal{D}(\Omega)$ consists of fn's φ defined on Ω, vanishing outside a b'nd set B of Ω s.t $\bar{B} \cap \partial\Omega = \emptyset$ & all the partial derivatives of all orders of φ are cts.
(i.e. $\varphi \in C^\infty$)

Note: $\mathcal{D}(\Omega)$ is a vector space.
Infact, it is a locally convex topological vector space.

eg: On $\mathbb{R}$,
$$\varphi(x) = \begin{cases} e^{-1/x^2}, & x > 0 \\ 0, & x \leq 0 \end{cases}$$

Note that $\dfrac{d^k}{dx^k} e^{-1/x^2} = \dfrac{p(x)}{q(x)} e^{-1/x^2}$



where p & q are polynomials.

1. $\varphi(x) = \psi(x)\psi(1-x)$ is a test fx^n & vanishes outside $0<x<1$

2. If φ is a test fx^n, then $\varphi_{x_0}(x) = \varphi(x+x_0)$ is also a test fx^n.

3. Sim., $a\varphi$, $a_1\varphi_1(x) + a_2\varphi_2(x)$ are test fx^ns.

4. Let $\Omega \subset \mathbb{R}$, $\varphi: \Omega \to \mathbb{R}$. Then with $\tilde{\Omega} = \Omega \times \Omega \times \dots \Omega \subseteq \mathbb{R}^n$, $\varphi_1: \tilde{\Omega} \to \mathbb{R}$ def. by $\varphi_1(x_1, \dots, x_n) = \varphi(x_1)\varphi(x_2)\cdots\varphi(x_n)$ is also a test fx^n.

Def^n: A seq. of test fx^ns $\{\varphi_n\}_{n=1}^{\infty}$ in $\mathcal{D}(\Omega)$ is said to converge to 0 in $\mathcal{D}(\Omega)$ if $\exists$ a compact set $K \subset \Omega$ s.t $\operatorname{supp} \varphi_m \subseteq K$ $\forall\, m \in \mathbb{Z}_{>0}$ & φ_m & all its derivatives converge uniformly to 0 on K.

$\underline{\text{Def}^n}$: The <u>class of distributions</u> on Ω, denoted by $\mathcal{D}'(\Omega)$ is defined as the set of all cts. linear functionals on $\mathcal{D}(\Omega)$.

The distributions will be denoted by $\langle f, \varphi \rangle$ where $\varphi \in \mathcal{D}(\Omega)$.

This means the linear functional is $\varphi \mapsto \langle f, \varphi \rangle$

By linearity, $\langle f, a_1\varphi_1 + a_2\varphi_2 \rangle = a_1 \langle f, \varphi_1 \rangle + a_2 \langle f, \varphi_2 \rangle$

eg: 1. If $f : \mathbb{R} \to \mathbb{R}$ is cts., then $\langle f, \varphi \rangle = \int f(x)\varphi(x)\, dx$ is a distribution

2. Since $\int_{|x|<r} |x|^{-t}\, dx$ converges for $t<n$ (in n dimensions)

& diverges for $t \geq n$.

The fn^n $|x|^{-t}$ for $t<n$ gives rise to the distribution

$$\langle f, \varphi \rangle = \int_{\mathbb{R}^n} |x|^{-t} \varphi(x)\, dx$$

$$\left(\int_{|x|<r} \frac{dx}{|x|^t} = \int\int_{|x|<r} \frac{dx_1\, dx_2}{(x_1^2 + x_2^2)^{t/2}} = \int_0^{2\pi}\int_0^{r} \frac{r\, dr\, d\theta}{r^t} = 2\pi \int_0^{r} \frac{dr}{r^{t-1}} \right.$$

$$\left. \text{converges for } 0 < t-1 < 1, \quad t < 2 = n \right)$$

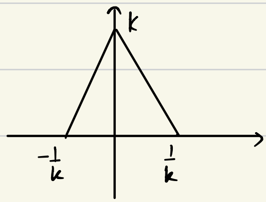
3. <u>Dirac δ-fxn</u> is defined by $\langle \delta, \varphi \rangle = \varphi(0)$

Linearity : $a_1 \langle \delta, \psi_1 \rangle + a_2 \langle \delta, \psi_2 \rangle = a_1 \psi_1(0) + a_2 \psi_2(0)$

$$= \langle \delta, a_1\psi_1 + a_2\psi_2 \rangle$$

where $\psi_1, \psi_2 \in \mathcal{D}(\Omega)$

<u>Motivation for δ</u> : Let f_k be given by



$$\langle f_k, \varphi \rangle = \int f_k(x)\varphi(x)\, dx$$

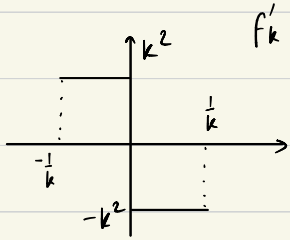
Assume $\varphi(x)$ to be const. around 0 for a small interval.

$$\Rightarrow \langle f_k, \varphi \rangle = \varphi(0) \int f_k(x)\, dx = \varphi(0)$$
$$= \langle \delta, \varphi \rangle$$

$\langle f_k, \varphi \rangle \to \langle \delta, \varphi \rangle$ as $k \to \infty$

i.e $f_k \to \delta$ in distribution

4. δ' is defined by $\langle \delta', \psi \rangle = -\psi'(0)$



$$\langle f_k', \varphi \rangle = \int f_k'(x)\varphi(x)\, dx$$
$$= \frac{k^2 \varphi(-1/2k)}{k} - \frac{k^2 \varphi(1/2k)}{k}$$

$$= \frac{\varphi(\frac{-1}{2k}) - \varphi(\frac{1}{2k})}{1/k}$$

$$= \frac{\varphi(\frac{-1}{2k}) - \varphi(0)}{1/k} + \frac{\varphi(0) - \varphi(\frac{1}{2k})}{1/k}$$

$$\lim_{k\to\infty} \langle f_k', \varphi \rangle = \lim_{k\to\infty} \frac{\varphi(\frac{-1}{2k}) - \varphi(0)}{1/k} + \frac{\varphi(0) - \varphi(\frac{1}{2k})}{1/k}$$

$$= -\frac{\varphi'(0)}{2} - \frac{\varphi'(0)}{2} = -\varphi'(0)$$

$\Rightarrow \quad \langle f_k', \varphi \rangle \to -\varphi'(0) = \langle \delta', \varphi \rangle$ as $k \to \infty$

$\Rightarrow \quad f_k' \to \delta'$ in distribution

Alternately, assuming f is ctsly diff, we get using integration by parts

$$\int f_k'(x)\,\varphi(x)\,dx = \overbrace{f_k(x)\,\varphi(x)\Big|_{-\infty}^{\infty}}^{0 \ (\because \varphi \text{ has compact support})} - \int f_k(x)\,\varphi'(x)\,dx$$

$$= -\int f_k(x)\,\varphi'(x)\,dx$$

$$= -\langle \delta, \varphi \rangle = -\varphi(0)$$

$$= \langle \delta', \varphi \rangle$$

<u>Def^n</u>: A fn^n f def. on Ω for which $\int f(x)\varphi(x)\,dx$ is absolutely conv. for every $\varphi \in \mathcal{D}(\Omega)$ is called <u>locally int'ble</u>, denoted by $f \in L^1_{loc}(\Omega)$.

eg: 1. Any int'ble fn^n is loc. int'ble.

2. $f(x) = 1$ is loc. int'ble but <u>not</u> int'ble on $\mathbb{R}$.

3. $f(x) = 1/x$ is loc. int'ble but <u>not</u> int'ble on $(0, \infty)$.

4. $f(x) = 1/x$ is <u>not</u> loc. int'ble on $\mathbb{R}$.

<u>Rem</u>: 1. If $f \notin L^1_{loc}(\Omega)$, then the associated distributions are badly behaved (i.e there can be more than one distribution).

For $\frac{1}{|x|}$, we have

$$\langle f_k, \varphi \rangle = \int_{-\infty}^{-a} \frac{\varphi(x)}{|x|}\,dx + \int_{-a}^{a} \frac{\varphi(x) - \varphi(0)}{|x|}\,dx + \int_{a}^{\infty} \frac{\varphi(x)}{|x|}\,dx$$

2. $\mathcal{D}(\Omega) \subset L^1_{loc}(\Omega) \subset \mathcal{D}'(\Omega)$

Def^n: We say a seq. $\{f_k\}_{k=1}^{\infty}$ of distr. conv. to a distr. f if the seq. $\{\langle f_k, \varphi \rangle\}_{k=1}^{\infty}$ conv. to $\langle f, \varphi \rangle$ $\forall \varphi \in \mathcal{D}(\Omega)$.

We write $f_k \to f$ in distribution.

Def^n: A operation means a linear operator, i.e.

1. for $\varphi \in \mathcal{D}(\Omega)$, $T\varphi \in \mathcal{D}(\Omega)$
2. $T(a_1\varphi_1 + a_2\varphi_2) = a_1 T\varphi_1 + a_2 T\varphi_2$

Def^n: Let f & g be fx^{n}s on $\mathbb{R}^n$. Then the convolution of f & g is defined as

$$(f * g)(x) = \int f(x-y)\, g(y)\, dy$$

whenever it exists.

Ex: $f * g = g * f$ for f, g int'ble.

For $\phi \in C^{\infty}(\mathbb{R})$ with compact support, $\int \phi(x)\, dx = 1$,
$\phi(x) \geq 0$, $\forall x \in \mathbb{R}$

Def: $\phi_\epsilon(x) = 1/\epsilon\, \phi(x/\epsilon) \quad \forall \epsilon > 0$.

Lem: If f is cts. on $\mathbb{R}$, then the fn $f * \phi_\epsilon$ conv. uniformly to f on every b'nd interval $[a,b]$.

Pf: Let $c > 0$ s.t $\phi(x) = 0 \quad \forall\, |x| \geq c$.

$$\int \phi_\epsilon(x)\, dx = \int \tfrac{1}{\epsilon}\, \phi\left(\tfrac{x}{\epsilon}\right) dx = \int \phi(z)\, dz = 1$$

$$\therefore \quad (f * \phi_\epsilon)(x) - f(x) = \int f(x-y)\, \phi_\epsilon(y)\, dy - \int f(x)\, \phi_\epsilon(y)\, dy$$

$$\left(\because \phi_\epsilon(x) = 0 \quad \forall\, |x| \geq c\epsilon\right) \qquad = \int_{-\epsilon c}^{\epsilon c} \left(f(x-y) - f(x)\right) \cdot \phi_\epsilon(y)\, dy$$

$$\therefore \quad |(f * \phi_\epsilon)(x) - f(x)| \leq \sup_{|y| \leq \epsilon c} |f(x-y) - f(x)| \int_{-\epsilon c}^{\epsilon c} \phi_\epsilon(y)\, dy$$

$$= \sup_{|y| \leq \epsilon c} |f(x-y) - f(x)|$$

Let $\epsilon' > 0$. The continuity of f gives that the fxn f is uniformly cts. on b'nd interval $[a,b]$

This implies $\exists\, \delta > 0$ s.t $x \in [a,b]$, $|y| < \delta \Rightarrow |f(x-y) - f(x)| < \epsilon'$

Choosing $\epsilon < \delta/c$, we get $|(f * \phi_\epsilon)(x) - f(x)| < \epsilon'$.

<u>Thm:</u> 1. Given any distr. $f \in \mathcal{D}'(\Omega)$, $\exists$ a seq. of test fx^{n}s $\{\varphi_n\}_{n=1}^{\infty}$ s.t $\varphi_n \to f$ in distr.

2. If $\{f_n\}_{n=1}^{\infty}$ is a seq. of distr. s.t $\lim_{n\to\infty} \langle f_n, \varphi \rangle$ exists $\forall\, \varphi \in \mathcal{D}(\Omega)$, then $\langle f, \varphi \rangle = \lim_{n\to\infty} \langle f_n, \varphi \rangle$ defines a distr. & $f_n \to f$ in distr.

(An application of UBP for locally convex topologies)

Def^n: If T is an operation, f is a distr. s.t $\{\varphi_n\}_{n=1}^{\infty}$ of test fn^{n}s with $\varphi_n \to f$ & $\lim_{n\to\infty} T\varphi_n$ exists & is indep. of the choice of $\{\varphi_n\}_{n=1}^{\infty}$, then Tf is defined as $\lim_{n\to\infty} T\varphi_n$

(This is the case for most interesting operators & it is a consequence of continuity)

Let $\varphi \in \mathcal{D}(\Omega)$, $\Omega \subseteq \mathbb{R}$.

Then by lemma, $(\varphi * \phi_\epsilon)(0) \to \varphi(0)$ as $\epsilon \to 0$

$\therefore$ $(\phi_\epsilon * \varphi)(0) \to \varphi(0)$ as $\epsilon \to 0$

$\Rightarrow \int \phi_\epsilon(-y)\varphi(y)\,dy \to \varphi(0)$

$\Rightarrow \int \tilde{\varphi}_n(z)\varphi(z)\,dz \to \varphi(0)$ $\left(\begin{array}{l}\text{with } \tilde{\varphi}_n(y) = \phi_{\frac{1}{n}}(-y) \\ \text{as } n\to\infty\end{array}\right)$

i.e $\langle \tilde{\varphi}_n, \varphi \rangle \to \langle \delta, \varphi \rangle$

i.e $\tilde{\varphi}_n \to \delta$ in distribution & $\int \tilde{\varphi}_n(x)\,dx = 1$

For $\tilde{\varphi}_n(x) = \phi_{1/n}(-x)$, we have $\tilde{\varphi}_n \to \delta$ in distribution

<u>Example 1</u> : Fix $y \in \mathbb{R}^n$.

Define $(T\varphi)(x) = \varphi(x+y)$, $\varphi \in \mathcal{D}(\mathbb{R}^n)$.

Let $\tilde{\psi}_n \in \mathcal{D}(\mathbb{R}^n)$ be s.t $\tilde{\psi}_n \to f$ in distr, i.e.

$$\langle f, \varphi \rangle = \lim_{n\to\infty} \langle \tilde{\psi}_n, \varphi \rangle \quad \forall\ \varphi \in \mathcal{D}(\mathbb{R}^n)$$

$$\begin{aligned}
\lim_{n\to\infty} \int T\tilde{\psi}_n(x)\, \varphi(x)\, dx &= \lim_{n\to\infty} \int \tilde{\psi}_n(x+y)\, \varphi(x)\, dx \\
&= \lim_{n\to\infty} \int \tilde{\psi}_n(z)\, \varphi(z-y)\, dz \qquad \begin{pmatrix} \text{with} \\ z = x+y \end{pmatrix} \\
&= \langle f, \varphi(\cdot - y) \rangle
\end{aligned}$$

So, $\langle Tf, \varphi \rangle = \langle f, \varphi(\cdot - y) \rangle$

We have $(\varphi * \phi_\epsilon)(-y) \to \varphi(-y)$

$$\begin{aligned}
&\Rightarrow (\phi_\epsilon * \varphi)(-y) \to \varphi(-y) \\
&\Rightarrow \int \phi_\epsilon(-y-z)\, \varphi(z)\, dz \to \varphi(-y) \quad \text{as } \epsilon \to 0 \\
&\Rightarrow \int \tilde{\varphi}_n(z+y)\, \varphi(z)\, dz \to \varphi(-y) \\
&\Rightarrow \int T\tilde{\varphi}_n(z)\, \varphi(z)\, dz \to \varphi(-y) \\
&\Rightarrow \langle T\delta, \varphi \rangle = \varphi(-y) \qquad \begin{pmatrix} \text{where } \tilde{\varphi}_n(x) = \phi_{1/n}(-x) \\ \tilde{\varphi}_n \to \delta \end{pmatrix}
\end{aligned}$$

Alternately, $\langle T\delta, \varphi \rangle = \langle \delta, \varphi(\cdot - y) \rangle = \varphi(-y)$

The distribution $\langle g, \varphi \rangle = \varphi(-y)$ is called the δ-fxn at $-y$.

Example 2 : Let $T = \frac{d}{dx}$ & $n=1$. We write $\tau_y(x) = \varphi(x+y)$ for translation & $I\varphi(x) = \varphi(x)$ for $\varphi \in \mathcal{D}(\mathbb{R})$

Then $$\frac{d}{dx} = \lim_{y \to 0} \frac{1}{y}(\tau_y - I)$$

i.e $$\frac{d}{dx} f = \lim_{y\to 0} \frac{\tau_y f - f}{y}$$

$$\langle \tau_y f, \varphi \rangle = \langle f, \tau_{-y} \varphi \rangle \qquad (\because \langle Tf, \varphi \rangle = \langle f, \varphi(\cdot - y) \rangle)$$

We have $$\left\langle \frac{d}{dx} f, \varphi \right\rangle = \lim_{y \to 0} \frac{1}{y} \left(\langle f, \tau_{-y}\varphi \rangle - \langle f, \varphi \rangle \right)$$
$$= \lim_{y\to 0} \left\langle f, \frac{1}{y}(\tau_{-y}\varphi - \varphi) \right\rangle$$

But, $$\lim_{y \to 0} \frac{1}{y} (\tau_{-y} \varphi - \varphi)(x) = \lim_{y\to 0} \frac{\varphi(x-y) - \varphi(x)}{y}$$
$$= -\varphi'(x)$$

$$\left\langle \frac{df}{dx}, \varphi \right\rangle = -\left\langle f, \frac{d\varphi}{dx} \right\rangle$$

Alternately, let $\{\tilde{\psi}_n\}_{n=1}^{\infty}$ be a seq. in $\mathcal{D}(\mathbb{R})$ s.t

$\tilde{\psi}_n \to f$ in distr.

$$\lim_{n\to\infty} \int \left(\frac{d}{dx} \tilde{\psi}_n(x)\right) \varphi(x)\, dx = \lim_{n\to\infty} - \int \tilde{\psi}_n(x)\, \varphi'(x)\, dx \quad \left(\begin{array}{c}\text{Integration} \\ \text{by Parts}\end{array}\right)$$

$$= -\langle f, \varphi' \rangle$$

So, $\left\langle \frac{df}{dx}, \varphi \right\rangle = -\langle f, \varphi' \rangle$

<u>Defⁿ</u>: A family of sets $\{E_i : i \in \mathcal{I}\}$ in $\mathbb{R}^n$ is said to be <u>locally finite</u> if for every pt. x, $\exists$ a nbd which intersects atmost finitely many E_i s.

<u>Thm</u>: Let Ω be an open set in $\mathbb{R}^n$ & let $\{\Omega_i : i \in \mathcal{I}\}$ be a family of open sets s.t $\Omega = \bigcup_{i \in \mathcal{I}} \Omega_i$.

Then $\exists$ C^∞ fxⁿs $\{\phi_i : i \in \mathcal{I}\}$ defined on Ω s.t

1. $\text{supp}\, \phi_i \subset \Omega_i$
2. $\{\text{supp}\, \phi_i : i \in \mathcal{I}\}$ is locally finite
3. $0 \leq \phi_i \leq 1 \quad \forall\, i \in \mathcal{I}$
4. $\sum_{i \in \mathcal{I}} \phi_i = 1$

The coll. $\{\phi_i : i \in \mathcal{I}\}$ is a <u>locally-finite C^∞ partition</u> of unity subordinate to the open cover $\{\Omega_i : i \in \mathcal{I}\}$

$$\left(\text{Uses} \quad f(x) = \begin{cases} e^{\frac{-a^2}{a^2 - |x|^2}}, & \text{if } |x| < a \\ 0, & \text{if } |x| \geq a \end{cases} \right)$$

Cor: Let K be compact set in $\mathbb{R}^n$.
Then $\exists\, \phi \in \mathcal{D}(\mathbb{R}^n)$ s.t $\phi = 1$ on K.

Pf: Let U be an open set whose closure is compact & $K \subset U$.
Then $\{U, \mathbb{R}^n \setminus K\}$ is an open cover of $\mathbb{R}^n$.

Let $\{\phi, \psi\}$ be a locally finite C^∞ partition of unity subordinate to this cover s.t $\operatorname{supp} \phi \subset U$, $\operatorname{supp} \psi \subset \mathbb{R}^n \setminus K$.

We have $0 \leq \phi, \psi$, $\quad \phi + \psi = 1$
Further $\psi \equiv 0$ on K. So, $\phi \equiv 1$ on K.

Also, $\operatorname{supp} \phi \subset U \subset \bar{U}$ & $\bar{U}$ is compact
Hence $\phi \in \mathcal{D}(\mathbb{R}^n)$.

The fx^n ϕ constructed above is called a <u>cutoff fx^n</u> wrt the compact set K.

<u>Claim</u> : δ cannot be generated by a $L^1_{loc}(\mathbb{R})$ fx^n.

<u>Pf</u> : Assume that $\exists\, f \in L^1_{loc}(\mathbb{R})$ s.t $\langle f, \varphi \rangle = \langle \delta, \varphi \rangle \;\forall\, \varphi \in \mathcal{D}(\mathbb{R})$

Then for every $\epsilon > 0$, $\exists$ a $\phi_\epsilon \in \mathcal{D}(\mathbb{R})$ with support in $B(0,\epsilon)$,

$0 \leq \phi_\epsilon \leq 1$ & $\phi_\epsilon = 1$ on $B(0,\epsilon/2)$

$$\langle \delta, \phi_\epsilon \rangle = \phi_\epsilon(0) = 1 \quad \forall\, \epsilon > 0$$

$$\langle \delta, \phi_\epsilon \rangle = \int f(x)\,\phi_\epsilon(x)\,dx = \int_{B(0,\epsilon)} f(x)\,\phi_\epsilon(x)\,dx$$

So, $$|\langle \delta, \phi_\epsilon \rangle| \leq \int_{B(0,\epsilon)} |f(x)|\,dx \to 0 \quad \text{as } \epsilon \to 0$$

This is a contradiction.

<u>Another example of distribution</u>

Let μ be a Borel measure s.t $\mu(K) < \infty$ for every compact $K \subseteq \mathbb{R}^n$.

Then $\phi \to \int \phi\, d\mu$, $\phi \in \mathcal{D}$ defines a distn.

Rem: 1. Distributions are infinitely differentiable

$$\left\langle \frac{d^n}{dx^n} f , \varphi \right\rangle = (-1)^n \left\langle f, \frac{d^n \varphi}{dx^n} \right\rangle \quad \forall\ n \in \mathbb{Z}_{>0}$$

$$\langle D^\alpha f, \varphi \rangle = (-1)^{|\alpha|} \langle f, D^\alpha \varphi \rangle$$

multinomial notation for partial derivatives

2. The distributional derivative & the distribution generated by the classical derivatives are not necessarily same.

Define Heavyside fxn H.

$$H(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

The derivative of H a.e is 0.

H is locally int'ble & hence defines a distr.

Let H' be the distr. derivative

$$\langle H', \varphi \rangle = -\langle H, \varphi' \rangle = -\int H(x) \varphi'(x)\, dx$$

$$= -\int_0^\infty \varphi'(x)\, dx = \varphi(0) = \langle \delta, \varphi \rangle \quad \forall\ \varphi \in \mathcal{D}$$

So, $H' = \delta$

For absolutely cts. fn's & C^∞ fn's, the derivatives are same.

<u>Rem</u> : Operations on distr.

- $\langle \tau_y f, \varphi \rangle = \langle f, \tau_{-y}\varphi \rangle \quad \forall \varphi \in \mathcal{D}$
- $\langle \frac{d}{dx} f, \varphi \rangle = -\langle f, \frac{d}{dx}\varphi \rangle \quad \forall \varphi \in \mathcal{D}$
- Linear $T(\mathcal{D}) \subseteq \mathcal{D}$, extending T to distributions

  i.e $\tilde{\varphi}_n \to f$ exists $\Rightarrow \lim_{n\to\infty} T\tilde{\varphi}_n$ exists & is indep. of $\tilde{\varphi}_n$

- Distributions are infinitely many times diff.
- Distr. derivative may <u>not</u> be same as the distr. induced by classical derivative.

  But they are same for absolutely cts. fn's & C^∞ fn's.

- A fn $f : [a,b] \to \mathbb{R}$ is said to be <u>abs. cts.</u> if $\forall \epsilon > 0$, $\exists \delta > 0$ s.t if $\{[a_i, b_i] : 1 \le i \le n\}$ is a coll. of disjoint intervals in $[a,b]$, then

$$\sum_{i=1}^{n} b_i - a_i < \delta \Rightarrow \sum_{i=1}^{n} |f(b_i) - f(a_i)| < \epsilon$$

- If f is abs. cts., then f is a.e differentiable.

Another method of defining operations on distributions is the <u>method of adjoint identities</u> which is as follows:

Let T be the op^n with another operation S s.t

$$\int T\psi(x)\,\varphi(x)\,dx = \int \psi(x)\,S\varphi(x)\,dx \quad , \quad \psi, \varphi \in \mathcal{D}$$

$$S\varphi \in \mathcal{D} \;\; \forall\, \varphi \in \mathcal{D}$$

Let $\varphi_n \to f$ in distr. where $\varphi_n \in \mathcal{D}$.

Then $$\lim_{n\to\infty} \int T\varphi_n(x)\,\varphi(x)\,dx = \lim_{n\to\infty} \int \varphi_n(x)\,S\varphi(x)\,dx$$

$$= \langle f, S\varphi \rangle$$

So, it is indep. of the choice of $\tilde{\varphi}_n \to f$

Thus, we can extend T to $\mathcal{D}'$ &

$$\langle Tf, \varphi \rangle = \langle f, S\varphi \rangle \quad \forall\, f \in \mathcal{D}', \varphi \in \mathcal{D}$$

In the previous two examples

$$\langle \tau_y f, \varphi \rangle = \langle f, \tau_{-y}\varphi \rangle$$

$$\& \;\; \langle \tfrac{d}{dx} f, \varphi \rangle = -\langle f, \tfrac{d}{dx}\varphi \rangle \quad \forall\, \varphi \in \mathcal{D}, f \in \mathcal{D}'$$

<u>Example</u>: Multipⁿ by a fxⁿ $m(x)$.

Clearly the adjoint identity is

$$\int (m(x)\psi(x))\,\varphi(x)\,dx = \int \psi(x)\,(m(x)\varphi(x))\,dx$$

To ensure, $m(x)\varphi(x) \in \mathcal{D}$ if $\varphi(x) \in \mathcal{D}$, we assume that $m(x)$ is infinitely diff.

So, if we denote this opⁿ by T, then

$$\langle Tf, \varphi \rangle = \langle f, m\varphi \rangle \quad \forall\, \varphi \in \mathcal{D}$$

Let m be a C^∞ fxⁿ. Then what is $m\delta'$?

For $\varphi \in \mathcal{D}$,

$$\langle m\delta', \varphi \rangle = \langle \delta', m\varphi \rangle = -\langle \delta, (m\varphi)' \rangle$$

$$= -\langle \delta, m'\varphi \rangle - \langle \delta, m\varphi' \rangle$$

$$= -m'(0)\varphi(0) - m(0)\varphi'(0)$$

Rem: 1. Even if $f \in L^1_{loc}$, the distr. $\left(\frac{\partial}{\partial x_n}\right) f$ need not correspond to a locally int'ble fxn.

Nevertheless, we can perform various opn's on this distr.

In this sense, distribution theory can be thought of as completion of differential calculus.

2. The product of two arbitrary distributions may not be defined.

We have $\operatorname{sgn} x = \begin{cases} 1 & \text{if } x>0 \\ 0 & \text{if } x=0 \\ -1 & \text{if } x<0 \end{cases}$

Consider, if $f(x) = \operatorname{sgn} x$, then

$$\langle f \cdot \delta, \varphi \rangle = \varphi(0) f(0) \quad \text{is a possibility.}$$

but there is no consistent way to define $f(0)$.

(classical derivative)

Thm: Let $f \in L^1_{loc}(\mathbb{R}^n)$, let $g = \frac{\partial}{\partial x_k} f$ exists & cts. except at a pt. y & let $g \in L^1_{loc}(\mathbb{R}^n)$. (define it arbitrarily at pt. y)

Then if $n \geq 2$, then the distr. derivative of f w.r.t x_k equals g, while for $n=1$, this is true if in addn, f is cts. at y.

Pf: Wlog let $y=0$

Case $n=1$: The distributional derivative

$$\left\langle \frac{d}{dx} f, \varphi \right\rangle = -\left\langle f, \frac{d}{dx}\varphi \right\rangle = -\int_{-\infty}^{\infty} f(x)\,\varphi'(x)\,dx$$

$$= -\int_{-\infty}^{0} f(x)\varphi'(x)\,dx - \int_{0}^{\infty} f(x)\varphi'(x)\,dx$$

$$= -\lim_{\epsilon \to 0} \left\{ \int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty} f(x)\varphi'(x)\,dx \right\} \qquad (\because f \in L^1_{loc})$$

$$= -\lim_{\epsilon \to 0} \Big\{ f(-\epsilon)\varphi(-\epsilon) - \int_{-\infty}^{-\epsilon} f'(x)\varphi(x)\,dx + f(\epsilon)\varphi(\epsilon) - \int_{\epsilon}^{\infty} f'(x)\varphi(x)\,dx \Big\}$$

$$= \{f(0)\varphi(0) - f(0)\varphi(0)\} + \lim_{\epsilon \to 0} \left\{ \int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty} f'(x)\varphi(x)\,dx \right\}$$

($\because f$ is cts at $y=0$)

$$= \lim_{\epsilon \to 0} \left\{ \int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty} g(x)\varphi(x)\,dx \right\}$$

$$= \int_{-\infty}^{\infty} g(x)\varphi(x)\,dx \qquad (\because g \in L^1_{loc})$$

Hence, the claim holds for this case.

Case $n \geq 2$: We would consider the $n=2$ case & $\frac{\partial}{\partial x_1}$

We have $\left(\frac{\partial}{\partial x_1} f, \varphi \right) = - \int f(x) \frac{\partial \varphi}{\partial x_1} dx$

$$= - \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(x_1, x_2) \frac{\partial \varphi}{\partial x_1}(x_1, x_2) \, dx_1 \right) dx_2$$

Now, for every $x_2 \neq 0$, $f(x_1, x_2)$ is ctsly diff. for all $x_1 \in \mathbb{R}$ & so for such x_2, we have

$$\int_{-\infty}^{\infty} f(x_1, x_2) \frac{\partial \varphi}{\partial x_1}(x_1, x_2) \, dx_1 = - \int_{-\infty}^{\infty} \frac{\partial f}{\partial x_1}(x_1, x_2) \varphi(x_1, x_2) \, dx_1 \quad \left(\begin{array}{c} \text{Int.} \\ \text{by} \\ \text{Parts} \end{array} \right)$$

$$= - \int_{-\infty}^{\infty} g(x_1, x_2) \varphi(x_1, x_2) \, dx_1$$

Substituting this back in the iterated integral, we get

$$\left(\frac{\partial f}{\partial x_1}, \varphi \right) = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} g(x_1, x_2) \varphi(x_1, x_2) \, dx_1 \right) dx_2$$

($\because$ The value of the inside integral at $x_2 = 0$ (single pt.) does not contribute to the double integral.)

$$= \int g(x) \varphi(x) \, dx$$

Hence, the claim holds for this case too.

Solving PDEs using distributions

Def^n: If a distribution is a sol^n of a diff. eq^n, then it is called a weak solution.

(1) Consider the 'vibrating string eq^n'

$$\frac{\partial^2 u}{\partial t^2}(x,t) = k^2 \frac{\partial^2 u}{\partial x^2}(x,t)$$

If f is twice ctsly diff., then $u(x,t) = f(x-kt)$ is a sol^n.

Claim: $u(x,t) = f(x-kt)$ for $f \in L^1_{loc}(\mathbb{R})$ is a weak sol^n.

Clearly, this defines a distr. For $\varphi \in \mathcal{D}$

$$\left\langle \frac{\partial^2 u}{\partial t^2}, \varphi \right\rangle = \left\langle u, \frac{\partial^2 \varphi}{\partial t^2} \right\rangle$$

$$\& \left\langle \frac{\partial^2 u}{\partial x^2}, \varphi \right\rangle = \left\langle u, \frac{\partial^2 \varphi}{\partial x^2} \right\rangle$$

So, $\left\langle \frac{\partial^2 u}{\partial t^2} - k^2 \frac{\partial^2 u}{\partial x^2}, \varphi \right\rangle = \left\langle u, \frac{\partial^2 \varphi}{\partial t^2} - k^2 \frac{\partial^2 \varphi}{\partial x^2} \right\rangle$

$$= \iint f(x-kt) \left(\frac{\partial^2 \varphi}{\partial t^2} - k^2 \frac{\partial^2 \varphi}{\partial x^2} \right) dx\, dt$$

We introduce new variables $y = x - kt$, $z = x + kt$.

$$\frac{\partial(y,z)}{\partial(x,t)} = \begin{bmatrix} \frac{\partial y}{\partial x} & \frac{\partial y}{\partial t} \\ \frac{\partial z}{\partial x} & \frac{\partial z}{\partial t} \end{bmatrix} = \begin{bmatrix} 1 & -k \\ 1 & k \end{bmatrix}$$

$$\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial t} \end{pmatrix} = \begin{pmatrix} 1 & -k \\ 1 & k \end{pmatrix}^T \begin{pmatrix} \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -k & k \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial y} \\ \frac{\partial}{\partial t} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \\ -k\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \end{pmatrix}$$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial y} + \frac{\partial}{\partial z}, \qquad \frac{\partial}{\partial t} = -k\frac{\partial}{\partial y} + k\frac{\partial}{\partial t}$$

$$\frac{\partial^2}{\partial t^2} = \left(-k\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\left(-k\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) = k^2\frac{\partial^2}{\partial y^2} - 2k^2\frac{\partial^2}{\partial z \partial y} + k^2\frac{\partial^2}{\partial z^2}$$

$$\frac{\partial^2}{\partial x^2} = \frac{\partial^2}{\partial y^2} + 2\frac{\partial^2}{\partial z \partial y} + \frac{\partial^2}{\partial z^2}$$

$$\Rightarrow \frac{\partial^2}{\partial t^2} - k^2 \frac{\partial^2}{\partial x^2} = -4k^2 \frac{\partial^2}{\partial y \partial z}$$

Thus, RHS becomes,

$$\iint f(x-kt) \left(\frac{\partial^2 \varphi}{\partial t^2} - k^2 \frac{\partial^2 \varphi}{\partial x^2} \right) dx\, dt = \iint f(y) \, -4k^2 \frac{\partial^2 \tilde{\varphi}(y,z)}{\partial y \partial z} \frac{dy\, dz}{2k}$$

$$= -2k \iint f(y) \frac{\partial^2 \tilde{\varphi}(y,z)}{\partial y \partial z} dy\, dz$$

$$= -2k \int f(y) \int \frac{\partial^2 \tilde{\varphi}(y,z)}{\partial y \partial z} dz\, dy$$

Since $\displaystyle\int_a^b \frac{\partial^2 \tilde{\varphi}(y,z)}{\partial y \partial z} dz = \frac{\partial \tilde{\varphi}}{\partial y}(y,b) - \frac{\partial \tilde{\varphi}}{\partial y}(y,a)$,

we get $\displaystyle\int_{-\infty}^{\infty} \frac{\partial^2 \tilde{\varphi}(y,z)}{\partial y \partial z} dz = 0$ $\left(\because \tilde{\varphi} \text{ is of compact support, so does } \frac{\partial \tilde{\varphi}}{\partial y} \right)$

Thus, we get $\displaystyle\left\langle \frac{\partial^2 u}{\partial t^2} - k^2 \frac{\partial^2 u}{\partial x^2}, \varphi \right\rangle = 0$

(2) Next, we would check if $u(x,y) = \log(x^2+y^2)$ is a weak sol^n for $\Delta u = 0$ (Laplace eq^n).

In the complement of $B(0,\epsilon)$, we have $\log r^2$ & it solves the eq^n.

$$\left\langle \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \varphi \right\rangle = \left\langle u, \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} \right\rangle \quad \text{for } \varphi \in \mathcal{D}$$

We use polar coordinates.

So, $\Delta u = \dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2}$ & $dx\,dy = r\,dr\,d\theta$

for $u = \log(x^2+y^2) = \log(r^2)$

We consider $\displaystyle\int_0^{2\pi}\int_0^{\infty} \log(r^2)\left(\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\varphi(r,\theta)\right) r\,dr\,d\theta$

To avoid the singularity of u at the origin, we integrate wrt r from ϵ to ∞ & let $\epsilon \to 0$

The computation is as follows

$$\int_0^{2\pi} (\log r^2) \frac{r}{r^2} \frac{\partial^2 \varphi(r,\theta)}{\partial \theta^2} \, d\theta = \frac{\log r^2}{r} \frac{\partial \varphi(r,\theta)}{\partial \theta} \Bigg|_0^{2\pi} = 0$$

because $\dfrac{\partial \varphi}{\partial \theta}$ is periodic.

Further, $\displaystyle\int_\epsilon^\infty \log(r^2) \frac{\partial}{\partial r} \varphi(r,\theta) \, dr$

$$= -\int_\epsilon^\infty \frac{\partial}{\partial r} \log(r^2) \cdot \varphi(r,\theta) \, dr - \log(\epsilon^2) \varphi(\epsilon,\theta)$$

$$\int_\epsilon^\infty (\log r^2) \, r \frac{\partial^2}{\partial r^2} \varphi(r,\theta) \, dr = -\int_\epsilon^\infty \frac{\partial}{\partial r} (\log(r^2) r) \frac{\partial \varphi(r,\theta)}{\partial r} \, dr$$

$$- \log(\epsilon^2) \epsilon \frac{\partial \varphi}{\partial r}(\epsilon,\theta)$$

$$= \int_\epsilon^\infty \frac{\partial^2}{\partial r^2} (\log(r^2) r) \varphi(r,\theta) \, dr - \log(\epsilon^2) \epsilon \frac{\partial \varphi}{\partial r}(\epsilon,\theta)$$

$$+ \frac{\partial}{\partial r} (\log(r^2) \cdot r) \, \varphi(r,\theta) \Bigg|_\epsilon^\infty$$

$$\frac{\partial}{\partial r} (r \log(r^2)) = \log(r^2) + \frac{2r^2}{r^2} = \log(r^2) + 2$$

$$\frac{\partial}{\partial r^2}(r\log(r^2)) = \frac{2}{r}$$

We get $\displaystyle\int_0^{2\pi}\int_\epsilon^\infty (\log r^2)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial \theta^2}\right)\varphi(r,\theta)\, r\, dr\, d\theta$

$$= \int_0^{2\pi}\int_\epsilon^\infty \left(-\frac{2}{r} + \frac{2}{r}\right)\varphi(r,\theta)\, dr + \int_0^{2\pi}\left(-\log(\epsilon^2) + \log(\epsilon^2) + 2\right)\frac{\partial\varphi(\epsilon,\theta)}{\partial r}\, d\theta$$

$$- \int_0^{2\pi}(\log\epsilon^2)\,\epsilon\, \frac{\partial\varphi(\epsilon,\theta)}{\partial r}\, d\theta$$

$$= 2\int_0^{2\pi}\varphi(\epsilon,\theta)\, d\theta + \int_0^{2\pi}\epsilon\log(\epsilon^2)\,\frac{\partial\varphi(\epsilon,\theta)}{\partial r}\, d\theta$$

$$\therefore \langle \Delta u, \varphi\rangle = \lim_{\epsilon\to 0}\left(2\int_0^{2\pi}\varphi(\epsilon,\theta)\, d\theta + \int_0^{2\pi}\epsilon\log(\epsilon^2)\,\frac{\partial\varphi(\epsilon,\theta)}{\partial r}\, d\theta\right)$$

$$= \lim_{\epsilon\to 0} 2\int_0^{2\pi}\varphi(\epsilon,\theta) - \varphi(0)\, d\theta + 4\pi\varphi(0) + \text{''}$$

$$= 0 + 4\pi\varphi(0) + 0 \quad \left(\because \frac{\partial\varphi}{\partial r} \text{ is bnd \& } \varphi \text{ is cts.}\right)$$

$$= 4\pi\varphi(0)$$

$$= 4\pi\langle\delta, \varphi\rangle$$

So, $\log(x^2+y^2)$ is <u>not</u> a weak sol^n of $\Delta u = 0$.

Rem: The computation shows that $u = \frac{1}{4\pi} \log(x^2+y^2)$ solves the eq^n $\Delta u = \delta$.

This is called the 'Fundamental sol^n of the Laplacian'.

Fourier Transform

If f is int'ble, then $\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)\, e^{-ix\xi}\, dx$

Under suitable cond^n, $f(x) \sim \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\xi)\, e^{ix\xi}\, d\xi$

Let F be 2π-periodic int'ble. Then $a_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(x)\, e^{-ikx}\, dx$

Def. $f\left(\frac{T}{2\pi} x\right) = F(x)$. So, f is T-periodic.

So, $a_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f\left(\frac{T}{2\pi} x\right) e^{-ikx}\, dx = \frac{1}{2\pi} \int_{-T/2}^{T/2} f(y) \cdot \frac{2\pi}{T} \cdot e^{-ik\frac{2\pi}{T}y}\, dy$ $\left(\text{with } y = \frac{Tx}{2\pi}\right)$

$$\Rightarrow a_k = \frac{1}{T} \int_{-T/2}^{T/2} f(y)\, e^{-ik\frac{2\pi}{T}y}\, dy$$

$$\Rightarrow Ta_k \sim g(\xi_k) = \int_{-T/2}^{T/2} f(y)\, e^{-i\xi_k y}\, dy$$

where $\xi_k = \frac{2\pi k}{T}$

So, as $T \to \infty$, $\quad g(\xi) = \int_{-\infty}^{\infty} f(y) e^{-i\xi y} dy$

$$f(x) \sim \sum_{k=-\infty}^{\infty} a_k e^{ikx} \Rightarrow f\left(\frac{T}{2\pi} x\right) \sim \sum_{k=-\infty}^{\infty} a_k e^{ik\frac{2\pi}{T} y}$$

$$\Rightarrow f(y) \sim \frac{1}{T} \sum_{k=-\infty}^{\infty} (T a_k) e^{i\xi_k y} = \frac{1}{T} \sum_{k=-\infty}^{\infty} g(\xi_k) e^{i\xi_k y}$$

$$= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} g(\xi_k) e^{i\xi_k y} \frac{2\pi}{T}$$

$$\Rightarrow f(y) \sim \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} g(\xi_k) e^{i\xi_k y} (\xi_k - \xi_{k-1})$$

As $T \to \infty$, $\quad f(y) \sim \frac{1}{2\pi} \int_{-\infty}^{\infty} \underbrace{g(\xi)}_{\hat{f}(\xi)} e^{i\xi y} d\xi$

$$\sum_{k=-\infty}^{\infty} |a_k|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 \, dx \qquad \text{(Parseval's identity)}$$

$$\Rightarrow \sum_{k=-\infty}^{\infty} |T \cdot a_k|^2 \cdot \frac{1}{T^2} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| f\left(\frac{T}{2\pi} x\right) \right|^2 dx$$

$$= \frac{1}{2\pi} \int_{-T/2}^{T/2} |f(y)|^2 \cdot \frac{2\pi}{T} \, dy$$

$$\Rightarrow \sum_{k=-\infty}^{\infty} |g(\xi_k)|^2 \cdot \frac{2\pi}{T^2} = 2\pi \int_{-T/2}^{T/2} |f(y)|^2 \, dy$$

$$\Rightarrow \sum_{k=-\infty}^{\infty} |g(\xi_k)|^2 (\xi_k - \xi_{k-1}) = 2\pi \int_{-T/2}^{T/2} |f(y)|^2 \, dy$$

As $T \to \infty$, $$\int_{-\infty}^{\infty} \underbrace{|g(\xi)|^2}_{\hat{f}} \, d\xi \sim 2\pi \int_{-\infty}^{\infty} |f(y)|^2 \, dy$$

(Plancherel's eq^n)

The Fourier series is equal to the given fn^n F, assuming F to be diff. (smooth).

But, for the Fourier transform, we would need smooth & decay at infinity.

Def^n: A fx^n f on $\mathbb{R}^n$ is said to be rapidly decreasing if $\forall N \in \mathbb{Z}_{\geq 0}$, $\exists M_N > 0$ s.t $|f(x)| \leq M_N |x|^{-N}$ as $|x| \to \infty$

A C^∞-fx^n is of Schwartz class $S(\mathbb{R}^n)$ if f & all its partial derivatives are rapidly decreasing.

eg: $\mathcal{D}(\mathbb{R}^n) \subseteq S(\mathbb{R}^n)$

Note that $e^{-|x|^2}$ for $x \in \mathbb{R}$ does not have compact support & so does not belong to $\mathcal{D}(\mathbb{R}^n)$. This is called the Gaussian.

But $f(x) = e^{-|x|^2} \in S(\mathbb{R}^n)$, because any partial derivative of $e^{-|x|^2}$ is of the form $P(x_1, \dots, x_n) e^{-|x|^2}$ where P is a polynomial & $e^{|x|^2}$ dominates all polynomials as $|x| \to \infty$

The properties of $S(\mathbb{R}^n)$ is as follows:

1. $S(\mathbb{R}^n)$ is a vector sp.

If $f, g \in S(\mathbb{R}^n)$, then for every $N \in \mathbb{Z}_{\geq 0}$, $\exists M_N, L_N$ s.t

$|f(x)| \leq M_N |x|^{-N}$, $|g(x)| \leq L_N |x|^{-N}$ as $|x| \to \infty$.

So, $|\ell f(x) + k g(x)| \leq (|\ell| M_N + |k| L_N) |x|^{-N}$ as $|x| \to \infty$

2. Products of f^n in $S(\mathbb{R}^n)$ also belong to $S(\mathbb{R}^n)$

$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)}$$ by Leibniz Rule in $\mathbb{R}$.

For $\mathbb{R}^n$, do the same for partial derivatives

3. $S(\mathbb{R}^n)$ is closed under multiplication by polynomials.

If P is a poly. of degree n, then for $f \in S(\mathbb{R}^n)$

$\exists\, M_{N+n} > 0$ s.t $|f(x)| \leq M_{N+n} |x|^{-(n+N)}$ as $|x| \to \infty$

Note, $|x_1 x_2| \leq |x|^2$. So, $|p(x)| \leq C|x|^n$

So, $|p(x) f(x)| \leq M_{N+n} C |x|^{-N}$ as $|x| \to \infty$

4. $S(\mathbb{R}^n)$ is closed under differentiation.

5. $S(\mathbb{R}^n)$ is closed under translation & multiplication by complex exponent $e^{ix\xi}$

Let $g(x) = f(x+a)$ where $f \in S(\mathbb{R}^n)$.

Then $\exists\, M_N > 0$ s.t $|g(x)| = |f(x+a)| \leq |x+a|^{-N}$ as $|x| \to \infty$

Since $\dfrac{|x+a|}{|x|} \to 1$ as $|x| \to \infty$, $\exists\, M_N' > 0$ s.t

$$M_N |x+a|^{-N} \leq M_N' |x|^{-N} \quad \text{as} \quad |x| \to \infty$$

$\therefore \quad |g(x)| \le M_N' |x|^{-N}$

If $h(x) = f(x) e^{ix\xi}$, then $|h(x)| = |f(x)|$

6. $S(\mathbb{R}^n)$ fn's are int'ble.

Let $f \in S(\mathbb{R}^n)$. So $\exists M_{n+1} > 0$ s.t $|f(x)| \le M_{n+1} |x|^{-(n+1)}$ as $|x| \to \infty$

Also, $\dfrac{|x|}{1+|x|} \to 1$ as $|x| \to \infty$

So, $\exists M > 0$ s.t $|f(x)| \le M(1+|x|)^{-(n+1)}$

$$\int (1+|x|)^{-(n+1)} dx = \int_{\varphi_{n-1}=0}^{2\pi} \int_{\varphi_{n-2}=0}^{\pi} \cdots \int_{\varphi_1=0}^{\pi} \int_{r=0}^{\infty} (1+r)^{-(n+1)} r^{(n-1)} \sin^{n-2}\varphi_1 \sin^{n-3}\varphi_2 \cdots \sin\varphi_{n-2} \, d\varphi_1 \, d\varphi_2 \ldots d\varphi_{n-1} \, dr$$

$$= C \int_0^\infty (1+r)^{-(n+1)} r^{(n-2)} dr < \infty$$

since $\displaystyle\int_1^\infty \frac{dr}{r^2} < \infty$. Use comparison test

$$\frac{(1+r)^{-(n+1)} r^{n-1}}{r^{-2}} = (1+r)^{-(n+1)} r^{n+1} = \left(1 + \frac{1}{r}\right)^{-(n-1)} \to 1 \text{ as } r \to \infty.$$

Def^n: The <u>Fourier transform</u> of $f \in S(\mathbb{R}^n)$ is defined by

$$F(f)(\xi) = \hat{f}(\xi) = \int f(x)\, e^{-ix\cdot\xi}\, dx \quad \text{for } x, \xi \in \mathbb{R}^n$$

where $x \cdot \xi = x_1\xi_1 + x_2\xi_2 + \dots + x_n\xi_n$

Pp^n: If $f \in S(\mathbb{R}^n)$, then

1. $F(\tau_y f)(\xi) = e^{iy\cdot\xi} F(f)(\xi)$ where $(\tau_y f)(x) = f(x+y)$

2. $F(e^{ix\cdot y} f(x))(\xi) = \tau_{-y} F(f)(\xi)$

3. $F(f(\delta x))(\xi) = \delta^{-n} F(f)(\delta^{-1}\xi)$ whenever $\delta > 0$

4. $\left(F \cdot \frac{\partial f}{\partial x_k}\right)(\xi) = i\xi_k F(f)(\xi)$

5. $\frac{\partial F}{\partial \xi_k} f(\xi) = -i F(x_k f(x))(\xi)$

Pf: 1. $\mathcal{F}(\tau_y f)(\xi) = \int f(x+y)\, e^{-ix\cdot\xi}\, dx$

$$= \int f(z)\, e^{-i(z-y)\cdot\xi}\, dz \qquad \text{with } z = x+y$$

$$= e^{iy\xi} \int f(z)\, e^{-iz\cdot\xi}\, dz = e^{iy\xi}\, \mathcal{F}(f)(\xi)$$

2. $\mathcal{F}(e^{ixy} f(x))(\xi) = \int f(x)\, e^{-ix(\xi-y)}\, dx$

$$= \mathcal{F}(f)(\xi-y) = \tau_y\, \mathcal{F}(f)(\xi)$$

3. $\mathcal{F}(f(\delta x))(\xi) = \int f(\delta x)\cdot e^{-ix\cdot\xi}\, dx$

$$= \int f(y)\, e^{-i\delta^{-1}y\cdot\xi}\, \delta^{-n}\, dy \qquad \left(\begin{array}{l} \text{with } y = \delta x \\ dy = \begin{vmatrix} \delta & & & \\ & \delta & & \\ & & \ddots & \\ & & & \delta \end{vmatrix} dx \\ \quad = \delta^{n}\, dx \end{array} \right)$$

$$= \delta^{-n}\, \mathcal{F}(f)(\delta^{-1}\xi)$$

4. $\mathcal{F}\left(\frac{\partial f}{\partial x_k}\right)(\xi) = \int \frac{\partial f}{\partial x_k}\, e^{-ix\cdot\xi}\, dx$

$$= \underbrace{f\, e^{-ix\cdot\xi}}_{0} \Big|_{-\infty}^{\infty} - \int f(x)(-i\xi_k)\, e^{-ix\cdot\xi}\, dx$$

$$= i\xi_k \int f(x)\, e^{-ix\cdot\xi}\, dx$$

$$= i\xi_k\, \mathcal{F}(f)(\xi)$$

5. Let $\epsilon > 0$. Let $e_k = (0, 0, \dots, 1, 0, \dots, 0)$
$\uparrow$ k^{th} pos.

$$\frac{F(f)(\xi + he_k) - F(f)(\xi)}{h} - (-i)F(x_k f(x))(\xi)$$

$$= \int f(x) e^{-ix\cdot\xi} \left(\frac{e^{-ix_k h} - 1}{h} + ix_k \right) dx$$

Since $f(x) \in S(\mathbb{R}^n)$, $x_k f(x)$ is rapidly decreasing, so
$\exists R > 0$ s.t $\displaystyle\int_{|x| \geq R} |x_k| |f(x)| \, dx < \epsilon/4$

$\left(\because \exists M > 0 \text{ s.t } |x_k f(x)| < \dfrac{M}{|x|^{n+1}}, \text{ so } \displaystyle\int |x_k f(x)| \, dx < \infty \right)$

Further for $|x| \leq R$, $\exists \delta > 0$ s.t $|h| < \delta \Rightarrow \left| \dfrac{e^{-ix_k h} - 1}{h} + ix_k \right| < \dfrac{\epsilon}{2VS}$

where $V = \displaystyle\int_{|x| \leq R} dx$ & $S = \sup |f(x)|$

By MVT, $\exists h$ s.t $0 < h_1 < h$ & $(e^{-ix_k h} - 1) = -ix_k h e^{-ix_k h_1}$
So, $\left| \dfrac{e^{-ix_k h} - 1}{h} \right| \leq |x_k|$

For $|h| < \delta$,

$$\left| \frac{\mathcal{F}(f)(\xi + he_k) - \mathcal{F}(f)(\xi)}{h} - (-i)\mathcal{F}(x_k f(x))(\xi) \right|$$

$$\leq \int_{|x| \leq R} \left| f(x) e^{-ix\cdot\xi} \left(\frac{e^{-ix_k h} - 1}{h} + ix_k \right) \right| dx + 2 \cdot \frac{\epsilon}{4}$$

$$\leq \frac{\epsilon}{2VS} \sup|f(x)| \left(\int_{|x| \leq R} dx \right) + \frac{\epsilon}{2} = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

So, $\dfrac{\partial}{\partial \xi_k} \mathcal{F}(f)(\xi) = -i\, \mathcal{F}(x_k f(x))(\xi)$

<u>Rem</u>: If f is int'ble, then $|\hat{f}(\xi)| \leq \int |f(x)|\, dx$ ie $\hat{f}$ is b'nd. If $f \in S(\mathbb{R}^n)$, then f is int'ble & hence $\hat{f}$ is b'nd.

Thm: If $f \in S(\mathbb{R}^n)$, then $\hat{f} \in S(\mathbb{R}^n)$

Pf: If $f \in S(\mathbb{R}^n)$, then $\hat{f}$ is C^∞

Claim: If $f \in S(\mathbb{R}^n)$, then $\hat{f}$ is rapidly decreasing.

It is enough to show that $p(\xi)\hat{f}(\xi)$ is b'nd for any polynomial $p(x)$.

But, $$p(\xi)\hat{f}(\xi) = F\Big(\underbrace{p\big(-i\tfrac{\partial}{\partial x}\big) f(x)}_{\in S(\mathbb{R}^n)}\Big)(\xi) \qquad (\text{by pp}^n (4))$$

(Here, $p(\frac{\partial}{\partial x})$ means we replace x_k by $\frac{\partial}{\partial x_k}$ in $p(x_1, \ldots, x_n)$)

Hence, by above remark $p(\xi)\hat{f}(\xi)$ is b'nd i.e $\hat{f}$ is rapidly decreasing

To show: Any derivative of $\hat{f}$ is also rapidly decreasing.

$$p(\partial/\partial\xi)\hat{f}(\xi) = F(p(-ix)f(x))(\xi) \qquad (\text{by pp}^n (5))$$

$\because$ $p(-ix)f(x) \in S(\mathbb{R}^n)$, using the Claim, it follows that $p(\frac{\partial}{\partial \xi})\hat{f}$ is rapidly decreasing

Hence, $\hat{f} \in S(\mathbb{R}^n)$.

Thm: $\mathcal{F}(e^{-\frac{|x|^2}{2}})(\xi) = (2\pi)^{n/2} e^{-|\xi|^2/2}$

Pf: Let $f(x) = e^{-t|x|^2}$ for $t>0$ & $x \in \mathbb{R}^n$

$$\hat{f}(\xi) = \int e^{-t|x|^2} \cdot e^{-ix\cdot\xi} \, dx$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-tx_1^2 - ix_1\xi_1} \cdot e^{-tx_2^2 - ix_2\xi_2} \cdot \ldots e^{-tx_n^2 - ix_n\xi_n} \, dx_1\, dx_2 \ldots dx_n$$

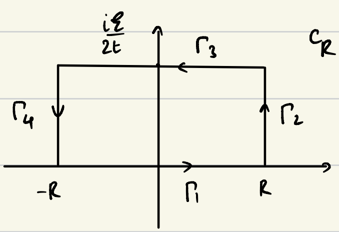
$$= \left(\int_{-\infty}^{\infty} e^{-tx_1^2 - ix_1\xi_1} \, dx_1\right) \cdots \left(\int_{-\infty}^{\infty} e^{-tx_n^2 - ix_n\xi_n} \, dx_n\right)$$

It is enough to compute $\int_{-\infty}^{\infty} e^{-tx^2 - ix\xi} \, dx$ for $x, \xi \in \mathbb{R}$

Now, $$\int_{-\infty}^{\infty} e^{-tx^2 - ix\xi} \, dx = \int_{-\infty}^{\infty} e^{-t\left(x + \frac{i\xi}{2t}\right)^2 - \frac{\xi^2}{4t^2}} \, dx$$

$$= e^{-\xi^2/4t^2} \int_{-\infty}^{\infty} e^{-t\left(x + \frac{i\xi}{2t}\right)^2} \, dx$$

We replace the above integral by $\int_{-\infty}^{\infty} e^{-tx^2} dx$ which is easy to compute.



For this, we would use Cauchy's Thm for the following contour C_R:

Since e^{-tz^2} is analytic,

$$\int_{C_R} e^{-tz^2} dz = 0 \qquad \text{(by Cauchy's thm)}$$

$$\Rightarrow \int_{\Gamma_1} e^{-tz^2} dz = \int_{-R}^{R} e^{-tx^2} dx$$

$$\int_{\Gamma_3} e^{-tz^2} dz = -\int_{-R}^{R} e^{-t\left(x+\frac{i\xi}{2t}\right)^2} dx$$

<u>To show</u> :

$$\left| \int_{\Gamma_2} e^{-tz^2} dz \right| = \left| \int_{\Gamma_2} e^{-t(x+iy)^2} dz \right|$$

$$= \left| \int_0^{\frac{\xi}{2t}} e^{-t(R^2-y^2) - 2itRy} dy \right|$$

$$\leq e^{-tR^2} \int_0^{\frac{\xi}{2t}} e^{ty^2} dy$$

$$\leq e^{-tR^2} \cdot e^{t\left(\frac{\xi}{2t}\right)^2} \cdot \frac{\xi}{2t} \to 0 \quad \text{as } R \to \infty$$

Sim. $\displaystyle\int_{\Gamma_4} e^{-tz^2} dz \to 0$ as $R \to \infty$

Thus, $$\int_{-\infty}^{\infty} e^{-tx^2} dx - \int_{-\infty}^{\infty} e^{-t\left(x+\frac{i\xi}{2t}\right)^2} dx = 0$$

$$\Rightarrow \int_{-\infty}^{\infty} e^{-t\left(x+\frac{i\xi}{2t}\right)^2} dx = \int_{-\infty}^{\infty} e^{-tx^2} dx$$

T'fore, $$\int_{-\infty}^{\infty} e^{-tx^2} e^{-ix\cdot\xi} dx = e^{-\frac{\xi^2}{4t}} \int_{-\infty}^{\infty} e^{-tx^2} dx$$

Claim : $$\int_{-\infty}^{\infty} e^{-tx^2} dx = \sqrt{\frac{\pi}{t}}$$

Pf : Let $$g(t) = \int_{-\infty}^{\infty} e^{-tx^2} dx$$

Then $$g^2(t) = \left(\int_{-\infty}^{\infty} e^{-tx^2} dx\right)\left(\int_{-\infty}^{\infty} e^{-ty^2} dy\right)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-t(x^2+y^2)} dx\, dy$$

$$= \int_{0}^{2\pi}\int_{0}^{\infty} e^{-tr^2}\, r \; dr \; d\theta$$

$$= \pi \int_{0}^{\infty} e^{-ty} dy = \pi \left.\frac{e^{-ty}}{(-t)}\right|_{0}^{\infty} = \frac{\pi}{t}$$

So, $$\int_{-\infty}^{\infty} e^{-tx^2} dx = \sqrt{\frac{\pi}{t}}$$

$$\therefore \int_{-\infty}^{\infty} e^{-tx^2} \cdot e^{-ix\xi}\, dx = e^{-\frac{\xi^2}{4t}} \left(\pi/t\right)^{1/2}$$

$$\therefore \int_{-\infty}^{\infty} e^{-t|x|^2} \cdot e^{-ix\cdot\xi}\, dx = (\pi/t)^{n/2}\, e^{-\frac{|\xi|^2}{4t}} \quad \text{for } \xi \in \mathbb{R}^n$$

For $t = 1/2$, $\displaystyle\int_{-\infty}^{\infty} e^{-\frac{1}{2}|x|^2} \cdot e^{-ix\cdot\xi}\, dx = (\pi/t)^{n/2}\, e^{-\frac{|\xi|^2}{2}}$

<u>Thm</u>: Let $K_\delta = \delta^{-n/2} e^{-\frac{\pi}{\delta}|x|^2}$. Then the coll. $\{K_\delta\}_{\delta>0}$ is a family of good kernels as $\delta \to 0$.

<u>Pf</u>: Recall, $\displaystyle\int_{\mathbb{R}^n} e^{-\pi|x|^2}\, dx = \left(\frac{\pi}{\pi}\right)^{n/2} = 1$ $\quad \left(\because \displaystyle\int_{\mathbb{R}^n} e^{-t|x|^2} dx = \left(\frac{\pi}{t}\right)^{n/2}\right)$

$$\therefore \int_{\mathbb{R}^n} K_\delta(x)\, dx = \int \delta^{-n/2} \cdot e^{-\pi|x|^2/\delta}\, dx = \delta^{-n/2} \left(\frac{\pi}{\pi/\delta}\right)^{n/2} = 1$$

So, ppt 1 of good kernel holds

Further, $K_\delta(x) \geq 0$, so ppt 2 of good kernels also holds.

For $\eta > 0$, $\displaystyle\int_{|x|>\eta} K_\delta(x)\,dx = \int_{|y|>\frac{\eta}{\delta^{n/2}}} e^{-\pi|y|^2}\,dy$ $\left(\begin{array}{l}\text{with } y_i = x_i/\delta^{1/2} \\ dy = \delta^{-n/2}\,dx\end{array}\right)$

$\to 0$ as $\delta \to 0$

So, ppt 3 of good kernels is also satisfied.

Note that if $f, g \in S(\mathbb{R}^n)$, then for any fixed $x \in \mathbb{R}^n$
$f(x-t)\,g(t) \in S(\mathbb{R}^n)$ & is thus rapidly decreasing

So, $\displaystyle\int_{\mathbb{R}^n} f(x-t)\,g(t)\,dt$ converges.
This integral is defined as $(f * g)$.

<u>Cor</u>: If $f \in S(\mathbb{R}^n)$, then $(f * K_n)(x) \to f(x)$ uniformly in x as $\delta \to 0$.

<u>Pf</u>: Let $\epsilon > 0$ be given. Since $f \in S(\mathbb{R}^n)$, $\exists\, B > 0$ s.t
$|f(z)| < B \quad \forall\, z \in \mathbb{R}^n$

Further $K > 0$. Since $f \in S(\mathbb{R}^n)$, $\exists\, R > 0$ s.t $|f(z)| < \epsilon/4 \quad \forall\, |z| \geq R$

Since f is cts. on $L = \{x : |x| \leq R+K\}$, it is uniformly cts.

So, we can find $\eta > 0$ s.t $\eta < K$ & $|f(x)-f(y)| < \epsilon/2$ whenever $|x-y| < \eta$, $x, y \in L$.

Observe that $|f(x)-f(y)| < \epsilon/2$ whenever $|x-y| < \eta$.
Thus, f is uniformly cts. on $\mathbb{R}^n$.

Now, $|(f * K_\delta)(x) - f(x)| = \left| \int f(x-t) K_\delta(t)\, dt - \int K_\delta(t) f(x)\, dt \right|$

(by ppt 1) $= \left| \int K_\delta(t) \left(f(x-t) - f(x)\right) dt \right|$

$(\because K_\delta \geqslant 0)$ $\leqslant \int_{|t|>\eta} + \int_{|t|<\eta} K_\delta(t) \, |f(x-t) - f(x)| \, dt$

By ppt 3, $\exists\, \delta' > 0$ s.t $\forall\ 0 < \delta < \delta'$, $\int_{|t|>\eta} K_\delta(t)\, dt < \frac{\epsilon}{4B}$

So, the first integral is $< 2B \cdot \epsilon/4B = \epsilon/2$

The second integral is $< \epsilon/2$ since f is uniformly cts.
& $\int K_\delta(t)\, dt = 1$

Hence, the corollary follows.

Ppn: (Multiplication formula)

If $f, g \in S(\mathbb{R}^n)$, then

$$\int_{\mathbb{R}^n} f(x)\cdot \hat{g}(x)\, dx = \int_{\mathbb{R}^n} \hat{f}(y)\cdot g(y)\, dy$$

A cts. (complex-valued) fn f on $\mathbb{R}^n$ is said to be of moderate decrease on $\mathbb{R}^n$ if $\exists\, M>0$ s.t $|f(x)| \leq \dfrac{M}{(1+|x|)^{n+1}}$ as $|x|\to\infty$

Suppose F is cts. on $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$ & of moderate decrease.

Then $F_1(x) = \displaystyle\int_{\mathbb{R}^n} F(x,y)\, dy$, $x\in\mathbb{R}^n$ is cts. & of moderate decrease.

Similarly, $F_2(y) = \displaystyle\int_{\mathbb{R}^n} F(x,y)\, dx$, $y\in\mathbb{R}^n$ is cts. & of moderate decrease.

Further $$\int_{\mathbb{R}^{2n}} F(x,y)\, dx\, dy = \int_{\mathbb{R}^n} F_1(x)\, dx = \int_{\mathbb{R}^n} F_2(y)\, dy$$

Rem:

- $\int_{-\infty}^{\infty} f(x)\, dx = \lim_{N\to\infty} \int_{-N}^{N} f(x)\, dx$

- $\mathcal{F}(e^{-t|x|^2}) = \left(\frac{\pi}{t}\right)^{n/2} e^{-\frac{|\xi|^2}{4t}}$

- $\{K_\delta(x)\}_{\delta>0}$, $K_\delta(x) = \delta^{-n/2} e^{-\frac{\pi}{\delta}|x|^2}$

- If $f \in S(\mathbb{R}^n)$, then $(f * K_\delta)(x) \to f(x)$ uniformly as $\delta \to 0$

Ex: If $f, g \in S(\mathbb{R}^n)$, then $f * g = g * f$

- For $h \in \mathbb{R}$, $\int_{-\infty}^{\infty} f(x+h)\, dx = \int_{-\infty}^{\infty} f(x)\, dx \quad \forall f \in S(\mathbb{R})$

(even for moderately decreasing f)

$$\left| \int_{-N}^{N} f(x+h)dx - \int_{-N}^{N} f(x)\, dx \right| = \left| \int_{-N-h}^{N-h} f(x)\, dx - \int_{-N}^{N} f(x)\, dx \right|$$

$$= \left| \int_{-N-h}^{-N} f(x)\, dx - \int_{N-h}^{N} f(x)\, dx \right|$$

$$\leq \left| \int_{-N-h}^{-N} f(x)\, dx \right| + \left| \int_{N-h}^{N} f(x)\, dx \right| \longrightarrow 0$$

as $N \to \infty$

– If F is cts. on $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$ & of moderate decrease, then

$$F_1(x) = \int_{\mathbb{R}^n} F(x,y)\, dy$$ is also cts. & of moderate decrease.

Sim. for $F_2(y) = \int_{\mathbb{R}^n} F(x,y)\, dx$

Further, $$\int_{\mathbb{R}^{2n}} F(x,y)\, dx\, dy = \int_{\mathbb{R}^n} F_1(x)\, dx = \int_{\mathbb{R}^n} F_2(y)\, dy \quad - (*)$$

<u>Ppn</u>: (Multiplication formula)

If $f, g \in S(\mathbb{R}^n)$, then

$$\int_{\mathbb{R}^n} f(x) \cdot \hat{g}(x)\, dx = \int_{\mathbb{R}^n} \hat{f}(y) \cdot g(y)\, dy$$

<u>Pf</u>: Let $F(x,y) = f(x)\, g(y)\, e^{-ixy}$, $x, y \in \mathbb{R}^n$

Then $$F_1(x) = \int f(x) g(y) e^{-ixy}\, dy = f(x) \cdot \hat{g}(x)$$
$$F_2(y) = \int f(x) g(y) e^{-ixy}\, dx = \hat{f}(y) \cdot g(y)$$

So, by $(*)$ we get $$\int_{\mathbb{R}^n} f(x)\, \hat{g}(x)\, dx = \int_{\mathbb{R}^n} \hat{f}(y)\, g(y)\, dy$$

Thm: (Fourier Inversion)

If $f \in S(\mathbb{R}^n)$, then $\quad f(x) = \frac{1}{(2\pi)^n} \int \hat{f}(\xi)\, e^{ix\xi}\, d\xi$

Pf: Claim: $\quad f(0) = \frac{1}{(2\pi)^n} \int \hat{f}(\xi)\, d\xi$

Let $\quad G_\delta(x) = e^{-\frac{\delta}{4\pi}|x|^2} \quad$ for $\delta > 0$

$$\hat{G}_\delta(\xi) = \int G_\delta(x)\, e^{-ix\xi}\, d\xi = \int e^{-\frac{\delta}{4\pi}|x|^2} e^{-ix\xi}\, d\xi$$

$$= \left(\frac{\pi \cdot 4\pi}{\delta}\right)^{n/2} e^{-\frac{4\pi}{4\delta}|\xi|^2} = (2\pi)^n \delta^{-n/2} e^{-\frac{\pi}{\delta}|\xi|^2}$$

$$= (2\pi)^n K_\delta(\xi)$$

By Multip^n formula, $\quad \int f(x) \cdot \hat{G}_\delta(x)\, dx = \int \hat{f}(\xi)\, G_\delta(\xi)\, d\xi$

i.e $\quad (2\pi)^n \int f(x)\, K_\delta(x)\, dx = \int \hat{f}(\xi)\, G_\delta(\xi)\, d\xi$

We can replace $K_\delta(x)$ by $K_\delta(-x)$ & since $\{K_\delta\}$ is a family of good kernels, we get as $\delta \to 0$, $\quad (2\pi)^n f(0) = \int \hat{f}(\xi)\, d\xi$

(Since $\quad \int f(x)\, K_\delta(-x)\, dx = (f * K_\delta)(0)$)

This proves the claim.

Let $F(y) = f(x+y)$ for $x, y \in \mathbb{R}^n$

$$(2\pi)^n f(x) = (2\pi)^n F(0) = \int \hat{F}(\xi)\, d\xi = \int \hat{f}(\xi)\, e^{ix\xi}\, d\xi$$

i.e $$f(x) = \frac{1}{(2\pi)^n} \int \hat{f}(\xi)\, e^{ix\xi}\, d\xi$$

Cor: The Fourier transform is a bijective mapping on the Schwartz space.

Pf: Let $$T(g)(x) = \frac{1}{(2\pi)^n} \int g(\xi)\, e^{ix\xi}\, d\xi \quad \text{for } g \in S(\mathbb{R}^n)$$

By prev. thm, we have $T \circ F = id$

$T \circ F(f) = f$ i.e $T \circ F = id$

$(\because\ F(f_1) = F(f_2) \Rightarrow T \circ F(f_1) = T \circ F(f_2) \Rightarrow f_1 = f_2$ (injectivity))

We need to show $F \circ T = id$ ($F \circ T(g) = g$ (surjectivity))

We would show $F \circ T = T \circ F$

Observe $\quad \mathcal{F}(f)(\xi) = \int f(x) e^{-ix\xi} dx$

$$= \frac{(2\pi)^n}{(2\pi)^n} \int f(x) e^{ix(-\xi)} dx$$

$$= (2\pi)^n (Tf)(-\xi) \qquad - (\#)$$

$\therefore \quad (\mathcal{F} \circ T)(f) = \frac{1}{(2\pi)^n} \mathcal{F}(\mathcal{F}(f(-x))) \qquad$ (by #)

$$\mathcal{F}(f(-x))(\xi) = \int f(-x) e^{-ix\xi} dx = \int f(y) e^{-iy(-\xi)} dy = \mathcal{F}(f)(\xi)$$

$$= \frac{1}{(2\pi)^n} \mathcal{F}((\mathcal{F}f)(-\xi))$$

$$= \frac{1}{(2\pi)^n} \widetilde{\mathcal{F}(\mathcal{F}f)} \qquad \left(\text{Here, } \tilde{k}(x) = k(-x)\right)$$

$$= T(\mathcal{F}f) \qquad (\text{by } \#)$$

Hence, $(\mathcal{F} \circ T)(f) = (T \circ \mathcal{F})(f)$

So, T is the inverse of $\mathcal{F}$. We use $\mathcal{F}^{-1}$ to denote T.

$\underline{\text{Pp}^{n}}$: If $f, g \in S(\mathbb{R}^n)$, then

1. $f * g \in S(\mathbb{R}^n)$
2. $\widehat{f * g} = \hat{f} * \hat{g}$

$\underline{\text{Pf}}$: 1) $\underline{\text{Claim}}$: For $l \in \mathbb{Z}$, we have $\sup\limits_{x} |x|^l |g(x-y)| \leqslant A_l (1+|y|)^l$ for some $A_l > 0$

$$
\begin{aligned}
|x|^l |g(x-y)| &= |y + (x-y)|^l |g(x-y)| \\
&\leqslant (|y| + |x-y|)^l |g(x-y)| \\
&= \sum_{i=0}^{l} \binom{l}{i} |y|^i |x-y|^{l-i} |g(x-y)| \\
&\leqslant \sum_{i=0}^{l} \binom{l}{i} |y|^i A_l \\
&= (1+|y|)^l A_l
\end{aligned}
$$

where $A_l = \max\{ |x|^j |g(x)| : j = 0, 1, \dots, l \}$

Hence, $\sup |x|^l |(f*g)(x)| = \sup |x|^l \left| \int f(y)\, g(x-y)\, dy \right|$
$$\leqslant A_l \int |f| (1+|y|)^l \, dy$$

So, $|x|^l |(f*g)(x)|$ is b'nd for every $l \geqslant 0$ $-(*)$

Since interchanging differentiation & integration is justified for Schwartz class fx^n s (use uniform conv.), we get

$$\left(\frac{\partial}{\partial x}\right)^{\alpha}(f*g)(x) = \left(\frac{\partial}{\partial x}\right)^{\alpha}\int f(y)\,g(x-y)\,dy$$

$$= \int f(y)\left(\frac{\partial}{\partial x}\right)^{\alpha} g(x-y)\,dy = f*\left(\frac{\partial}{\partial x}\right)^{\alpha} g$$

Hence, $(*)$ is true for derivatives of $f*g$ i.e $f*g \in S(\mathbb{R}^n)$

2) Let $F(x,y) = f(y)\,g(x-y)\,e^{-ix\xi}$

So, $F_1(x) = \int_{\mathbb{R}^n} F(x,y)\,dy = \int f(y)\,g(x-y)\,e^{-ix\xi}\,dy$

$$= e^{-ix\xi}(f*g)(x)$$

& $F_2(y) = \int_{\mathbb{R}^n} f(y)\,g(x-y)\,e^{-ix\xi}\,dx = f(y)\,e^{-iy\xi}\,\hat{g}(\xi)$

Also, $\int F_1(x)\,dx = \int F_2(y)\,dy$

i.e $\widehat{f*g}(\xi) = \hat{f}(\xi)\,\hat{g}(\xi)$

Let $\langle f, g \rangle = \int f(x)\overline{g(x)}\,dx$

$\|f\|^2 = \langle f, f \rangle$

Thm: (Plancherel)

If $f \in S(\mathbb{R}^n)$, then $\|f\| = \frac{1}{(2\pi)^{n/2}} \|\hat{f}\|$

Pf: Let $f^b(x) = \overline{f(-x)}$. Let $h(x) = (f * f^b)(x)$

$$\hat{f^b}(\xi) = \int f^b(x) e^{-ix\xi} dx = \int \overline{f(-x)}\, e^{-ix\xi}\, dx$$

$$= \overline{\int f(-x)\, e^{ix\xi} dx}$$

$$= \overline{\int f(y)\, e^{-iy\xi} dy}$$

$$= \overline{\hat{f}(\xi)}$$

$$\hat{h}(\xi) = (f \hat{*} f^b)(\xi) = \hat{f}(\xi) \cdot \hat{f^b}(\xi) = \hat{f}(\xi) \cdot \overline{\hat{f}(\xi)} = |\hat{f}(\xi)|^2$$

$$h(0) = \int f(y) \cdot f^b(-y)\, dy = \int f(y)\, \overline{f(y)}\, dy = \int |f(y)|^2 dy$$

$$\int |f(y)|^2 dy = h(0) = \frac{1}{(2\pi)^n} \int \hat{h}(\xi)\, d\xi = \frac{1}{(2\pi)^n} \int |\hat{f}(\xi)|^2 d\xi$$

$$\Rightarrow \|f\| = \frac{1}{(2\pi)^{n/2}} \|\hat{f}\|$$

Thm: If $f, g \in S(\mathbb{R}^n)$, then $F(fg) = \frac{1}{(2\pi)^n} \hat{f} * \hat{g}$

Pf: For $g \in S(\mathbb{R}^n)$, $F^2(g)(\xi) = (2\pi)^n \tilde{g}(\xi)$

$\left(\text{Recall, } \frac{1}{(2\pi)^n} \widetilde{F^2(g)} = T \circ F(g) = g \right)$

We have $F(f * g) = F(f) \cdot F(g)$

Replacing f & g by $F(f)$ & $F(g)$, we obtain,

$$F(F(f) * F(g)) = F^2(f) \cdot F^2(g) = (2\pi)^{2n} \tilde{f} \tilde{g} = (2\pi)^n F^2(fg)$$

By applying F^{-1} to both the sides, we get $F(f) * F(g) = (2\pi)^n F(fg)$

Fourier Transforms of Tempered Distributions

Defⁿ: Cts. linear functionals on the space of Schwartz class fx^ns are called Tempered distributions.

The class of tempered distributions is denoted by $S'(\mathbb{R}^n)$.

eg: For int'ble fx^ns, then tempered distribution is

$$\langle f, \varphi \rangle = \int f(x)\varphi(x)\,dx \quad \text{where } \varphi \in S(\mathbb{R}^n)$$

Rem: Every distr. is not a tempered distr.

Let $f(x) = e^{x^2}$ for $x \in \mathbb{R}$.

Clearly, f defines a distr. $\int f(x)\varphi(x)\,dx = \int e^{x^2}\varphi(x)\,dx < \infty$ for $\varphi \in \mathcal{D}$

Note that $e^{-x^2/2} \in S(\mathbb{R})$ & $\int f(x) e^{-x^2/2}\,dx = \int e^{x^2} \cdot e^{-x^2/2}\,dx$

$$= \int e^{x^2/2}\,dx = \infty$$

($\because$ $e^{x^2/2}$ dominates x^2 as $|x| \to \infty$ & $\int_{-\infty}^{\infty} x^2\,dx = \infty$)

Ex: If $f \in L^1_{loc}$ & $\exists\, C > 0$ & $N \in \mathbb{Z}_{>0}$ s.t $\int_{|x|<A} |f(x)|\,dx \leq CA^N$ as $A \to \infty$, then f defines a tempered distr.

eg: Polynomials

Adjoint identity for $S'(\mathbb{R})$

If T & S are operations that maps $S(\mathbb{R}^n)$ to $S(\mathbb{R}^n)$ &

$$\int (T\psi(x))\,\varphi(x)\,dx = \int \psi(x)\,S\varphi(x)\,dx \quad \text{for} \quad \psi, \varphi \in S(\mathbb{R}^n)$$

then we define $\langle Tf, \varphi \rangle = \langle f, S\varphi \rangle$ for $f \in S'(\mathbb{R})$, $\varphi \in S(\mathbb{R}^n)$

The concept of tempered distr. is introduced to deal with Fourier transforms, because Fourier transforms preserve Schwartz class fx^n's but <u>not</u> $\mathcal{D}'(\Omega)$.

All operations we discussed in the prev. section on $\mathcal{D}'(\mathbb{R}^n)$ are also valid on $S'(\mathbb{R}^n)$ except 'multip^n by C^∞-fx^n $m(x)$'.

Multip^n by C^∞-fx^n $m(x)$ defines an operation on $S'(\mathbb{R}^n)$ if $|m(x)| \leq C|x|^N$ as $|x| \to \infty$

Especially $m(x)$ cannot be $e^{|x|^2}$ because $e^{|x|}e^{-|x|^2/2} \notin S(\mathbb{R}^n)$ though $e^{-|x|^2/2} \in S(\mathbb{R}^n)$

Recall that for $\psi, \varphi \in S(\mathbb{R}^n)$, we have

$$\int \hat{\psi}(x)\varphi(x)\,dx = \int \psi(x)\,\hat{\varphi}(x)\,dx$$

- Distributions ($\mathcal{D}'(\mathbb{R}^n)$) v/s Tempered distributions ($S'(\mathbb{R}^n)$)
- Topology on $S'(\mathbb{R}^n)$: A seq. $\{\varphi_n\}_{n=1}^{\infty}$ in $S(\mathbb{R}^n)$ is said to conv. to 0 if for every polynomial p & every differential operator L with const. coeff., then seq. $p(x) L \varphi_n(x) \to 0$ uniformly as $n \to \infty$.

Rem : Inclusion map $\iota : \mathcal{D}(\mathbb{R}^n) \hookrightarrow S(\mathbb{R}^n)$ is cts.

The space $\mathcal{D}(\mathbb{R}^n)$ is dense in $S(\mathbb{R}^n)$ (check) – (*)

Given any $f \in S'(\mathbb{R}^n)$, the fn $f\big|_{\mathcal{D}(\mathbb{R}^n)} \in \mathcal{D}'(\mathbb{R}^n)$

Because of (*), different elems. of $S'(\mathbb{R}^n)$ define different elems. of $\mathcal{D}'(\mathbb{R}^n)$.

Infact, $S'(\mathbb{R}^n)$ is a subspace of $\mathcal{D}'(\mathbb{R}^n)$

- We have the adjoint identity : For $\varphi, \psi \in S'(\mathbb{R}^n)$

$$\int \hat{\psi}(x)\varphi(x)\,dx = \int \psi(x)\hat{\varphi}(x)\,dx$$

$$\& \quad F : S(\mathbb{R}^n) \to S(\mathbb{R}^n)$$

So, for $f \in S'(\mathbb{R}^n)$, $\langle F(f), \varphi\rangle = \langle \hat{f}, \varphi\rangle = \langle f, \hat{\varphi}\rangle$ for $\varphi \in S(\mathbb{R}^n)$

Rem: If f is int'ble, then $\hat{f}$ is (uniformly) cts.

Let $\epsilon > 0$. Since f is int'ble, $\exists\, N$ s.t $\displaystyle\int_{|x|>N} |f(x)|\, dx < \epsilon$

for $|x| < N$, $\quad |e^{-ix(\xi+h)} - e^{-ix\xi}| = |e^{-ixh} - 1| \leqslant |h|\, N$ (using MVT)

So, with $\delta = \epsilon/N$, we get $|h| < \delta \Rightarrow |e^{-ix(\xi+h)} - e^{-ix\xi}| < \epsilon$

So, for $|h| < \delta$, we have

$$|\hat{f}(\xi+h) - \hat{f}(\xi)| = \left| \int f(x)\, e^{-ix(\xi+h)} dx - \int f(x)\, e^{-ix\xi} dx \right|$$

$$= \left| \int f(x) \left(e^{-ix(\xi+h)} - e^{-ix\xi}\right) dx \right|$$

$$\leqslant \int_{|x|\leqslant N} + \int_{|x|>N} \left(|f(x)|\, |e^{-ix(\xi+h)} - e^{-ix\xi}|\right) dx$$

$$< \epsilon \int |f(x)|\, dx + 2\epsilon = c\epsilon$$

T'fore $\hat{f}$ is uniformly cts.

<u>Claim</u> : The definition of Fourier transform for tempered distributions is consistent with that for fn's.

<u>Pf</u> : Since $\mathcal{D}(\mathbb{R}^n)$ is dense in the space of all int'ble fn's, we have $S(\mathbb{R}^n)$ is dense in the space of all int'ble fn's,

Let f be int'ble. So $\exists$ seq. $\{\varphi_n\}_{n=1}^{\infty}$ in $S(\mathbb{R}^n)$ s.t

$$\int |\varphi_n(x) - f(x)| \, dx \to 0 \quad \text{as} \quad n \to \infty$$

Also, for any int'ble fn g, we have $|\hat{g}(x)| \leq \int |g(x)| \, dx$

Since $|(\hat{\varphi}_n - \hat{f})(\xi)| \leq \int |\varphi_n - f| \, dx \quad \forall x$, we get $\hat{\varphi}_n \to \hat{f}$ uniformly

Hence, $\int \hat{\varphi}_n(x) \psi(x) \, dx \to \int \hat{f}(x) \psi(x) \, dx \quad - (i)$

i.e $\int \varphi_n(x) \hat{\psi}(x) \, dx \to \int \hat{f}(x) \psi(x) \, dx$

Also, $\int \varphi_n(x) \hat{\psi}(x) \, dx \to \int f(x) \hat{\psi}(x) \, dx \quad - (ii)$

$$\left(\left| \int (\varphi_n(x) - f(x)) \hat{\psi}(x) \, dx \right| \leq \sup \|\hat{\psi}\| \int |\varphi_n(x) - f(x)| \, dx \right)$$

From (i) & (ii), $\int \hat{f}(x) \psi(x) \, dx = \int f(x) \hat{\psi}(x) \, dx$

This proves the claim.

Riemann-Lebesgue Lemma: If f is int'ble, then $\lim_{|\xi|\to\infty} |\hat{f}(\xi)| = 0$

Pf: (continuing from prev. argument)

Let $\epsilon > 0$. So, $\exists\, n_0 \in \mathbb{Z}_{>0}$ s.t $|\hat{\varphi}_n(\xi) - \hat{f}(\xi)| < \epsilon/2 \quad \forall n \geq n_0, \forall \xi$

Further, for $\hat{\varphi}_{n_0}$, $\exists\, N > 0$ s.t for $|\xi| > N$, $|\hat{\varphi}_{n_0}(\xi)| < \epsilon/2$

Thus, for $|\xi| > N$, $|\hat{f}(\xi)| \leq |\hat{\varphi}_{n_0}(\xi) - \hat{f}(\xi)| + |\hat{\varphi}_{n_0}(\xi)| < \epsilon$

Note: $f \in L^1(\mathbb{R}^n) \Rightarrow \hat{f} \in C_0(\mathbb{R}^n)$

Thm: (Fourier Inversion Formula)

The inverse of distributional Fourier transform is given by

$$\mathcal{F}^{-1} f = (2\pi)^{-n} (\mathcal{F} f)^{\sim}$$

(where for $f \in S'(\mathbb{R}^n)$, $\langle \tilde{f}, \varphi \rangle = \langle f, \tilde{\varphi} \rangle$ & $\tilde{\varphi}(x) = \varphi(-x)$ for $\varphi \in S(\mathbb{R}^n)$)

Pf: Recall that $\mathcal{F}^{-1}\varphi = (2\pi)^{-n}(\mathcal{F}\varphi)^{\sim}$ $\forall \varphi \in S(\mathbb{R}^n)$

Let $Sf = (2\pi)^{-n}(\mathcal{F}f)^{\sim}$ for $f \in S'(\mathbb{R}^n)$

So, for $f \in S'(\mathbb{R}^n)$, $\varphi \in S(\mathbb{R}^n)$

$$\begin{aligned}
\langle f, \varphi \rangle = \langle f, \mathcal{F}\mathcal{F}^{-1}\varphi \rangle &= \langle \mathcal{F}f, \mathcal{F}^{-1}\varphi \rangle \\
&= (2\pi)^{-n} \langle \mathcal{F}f, (\mathcal{F}\varphi)^{\sim} \rangle \\
&= (2\pi)^{-n} \langle (\mathcal{F}f)^{\sim}, \mathcal{F}\varphi \rangle \\
&= \langle Sf, \mathcal{F}\varphi \rangle = \langle \mathcal{F}Sf, \varphi \rangle
\end{aligned}$$

So, $\mathcal{F}Sf = f$ & similarly $S\mathcal{F}f = f$.

Hence, the formula holds.

Example 1: Consider distribution δ. Then $\varphi \in S(\mathbb{R}^n)$

$$\langle \hat{\delta}, \varphi \rangle = \langle \delta, \hat{\varphi} \rangle = \hat{\varphi}(0) = \int \varphi(x)\, dx$$

$\therefore$ $\hat{\delta} = 1$

Thm: $\mathcal{F}\left(\frac{\partial}{\partial x_k} f\right) = i\xi_k \hat{f}$ for $f \in S'(\mathbb{R}^n)$

Pf: $\left\langle \mathcal{F}\left(\frac{\partial}{\partial x_k} f\right), \varphi \right\rangle = \left\langle \frac{\partial f}{\partial x_k}, \hat{\varphi} \right\rangle$ for $\varphi \in S(\mathbb{R}^n)$

$$= - \left\langle f, \frac{\partial \hat{\varphi}}{\partial x_k} \right\rangle$$

$$= - \langle f, \mathcal{F}(-i\xi_k \varphi) \rangle$$

$$= \langle \mathcal{F}(f), i\xi_k \varphi \rangle$$

$$= \langle i\xi_k \mathcal{F}(f), \varphi \rangle$$

Hence, $\mathcal{F}\left(\frac{\partial}{\partial x_k} f\right) = i\xi_k \hat{f}$

Formulas similar to those of Fourier transform of $S(\mathbb{R}^n)$ also hold for Fourier transform of tempered distr.

Rem:

- f is smooth $\Rightarrow$ $\hat{f}$ has rapid decay at ∞
  f has rapid decay at ∞ $\Rightarrow$ $\hat{f}$ is smooth

- Since δ is <u>not</u> smooth, so $\hat{\delta}$ has no decay at ∞.
  But δ has rapid decay at infinity, so $\hat{\delta}$ is smooth.

Example 2 : Consider distribution δ'.

Then $\hat{\delta'} = i\xi\hat{\delta} = i\xi$

Since δ' is rougher than δ, the Fourier transform of δ' grows at ∞.

Example 3 : Let $f(x) = e^{is|x|^2}$ for $x \in \mathbb{R}^n$, $s \neq 0$ real.

Recall $F(e^{-t|x|^2}) = \left(\frac{\pi}{t}\right)^{n/2} e^{-|\xi|^2/4t}$

$$\therefore \int e^{-t|x|^2} \hat{\varphi}(x)\, dx = \left(\frac{\pi}{t}\right)^{n/2} \int e^{-|\xi|^2/4t} \varphi(\xi)\, d\xi \quad \text{for } \varphi \in S(\mathbb{R}^n)$$

We consider the analytic continuation for $\operatorname{Re} z > 0$ & obtain $F(z) = \int e^{-z|x|^2} \hat{\varphi}(x)\, dx$,

$$G(z) = \left(\frac{\pi}{t}\right)^{n/2} \int e^{-|\xi|^2/4z} \varphi(\xi)\, d\xi$$

Note that for $z = a + ib$ with $a > 0$.

So, $e^{-z|x|^2} \hat{\varphi}(x) = e^{-(a+ib)|x|^2} \hat{\varphi}(x)$

$$\operatorname{Re}\left(\frac{1}{z}\right) = \operatorname{Re}\left(\frac{1}{a+ib}\right) = \operatorname{Re}\left(\frac{a-ib}{a^2+b^2}\right) > 0$$

But F & G are equal if z is real.

So, by Identity Principle, $F(z) = G(z) \ \forall \ Re(z) > 0$

Finally for $s \neq 0$,

$$F(is) = \lim_{\epsilon \to 0^+} F(\epsilon + is) \quad \left(\begin{array}{l} \because F \ \& \ G \text{ are} \\ \text{cts. up to the b'ry} \end{array} \right)$$
$$= \lim_{\epsilon \to 0^+} G(\epsilon + is)$$
$$= G(is)$$

i.e. $$\int e^{is|x|^2} \hat{\varphi}(x)\, dx = \left(\frac{\pi}{is}\right)^{n/2} \int e^{-|\xi|^2/4(is)} \varphi(\xi)\, d\xi$$

The square root appearing in $\left(\frac{-\pi}{is}\right)^{n/2}$ can be uniquely determined in the following way.

For $Re(z) \geqslant 0$, we take the sq. root $z^{1/2}$ by requiring $\arg(z)$ to satisfy $-\frac{\pi}{2} \leqslant \arg(z) \leqslant \frac{\pi}{2}$

(This is consistent with choosing the positive sq. root when z is real & positive)

So, $\left(\frac{-\pi}{is}\right)$ becomes $\left(\frac{i\pi}{|s|}\right)$ when $s > 0$ & becomes $\left(\frac{-i\pi}{|s|}\right)$ when $s < 0$.

Hence,
$$\left(\frac{-\pi}{is}\right)^{n/2} = \begin{cases} \left(\frac{\pi}{|s|}\right)^{n/2} e^{\frac{in\pi}{4}}, & s > 0 \\ \left(\frac{\pi}{|s|}\right)^{n/2} e^{-\frac{in\pi}{4}}, & s < 0 \end{cases}$$

So, $$F(e^{is|x|^2}) = \left(\frac{-\pi}{is}\right)^{n/2} e^{\frac{|\xi|^2}{4is}}$$

Example 4: Let $f(x) = e^{-t|x|}$, $t > 0$

This is rapidly decreasing fx^n but it is not in $S(\mathbb{R}^n)$ because it fails to be diff. at $x = 0$.

for $n=1$,
$$\hat{f}(\xi) = \int_{-\infty}^{\infty} e^{-t|x|} e^{-ix\cdot\xi} dx$$
$$= \int_{-\infty}^{0} e^{tx - ix\xi} dx + \int_{0}^{\infty} e^{-tx - ix\xi} dx$$
$$= \left.\frac{e^{x(t-i\xi)}}{t-i\xi}\right|_{-\infty}^{0} + \left.\frac{e^{x(-t-i\xi)}}{-t-i\xi}\right|_{0}^{\infty}$$
$$= \frac{1}{t-i\xi} - \frac{1}{-t-i\xi} = \frac{2t}{t^2+\xi^2}$$

$\therefore$ By Fourier Inversion formula,
$$e^{-t|x|} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{t}{t^2+\xi^2} e^{ix\xi} d\xi \quad -(*)$$

for $n > 1$, we attempt to write $e^{-t|x|}$ as an 'average' of Gaussian $e^{-s|x|^2}$ because we know that the Fourier transform of $e^{-s|x|^2}$ i.e, we want to find $g(s,t)$ s.t
$$e^{-t|x|} = \int_{0}^{\infty} g(s,t)\, e^{-\frac{|x|^2}{4s}} ds$$

Note that $\displaystyle\int_0^\infty e^{-st^2} e^{-s\xi^2}\, ds = \left.\frac{e^{-s(t^2+\xi^2)}}{-(t^2+\xi^2)}\right|_0^\infty = \frac{1}{t^2+\xi^2}$

Substituting this in $(*)$, we get

$$e^{-t|x|} = \frac{1}{\pi}\int_{-\infty}^{\infty} t \int_0^\infty e^{-st^2} e^{-s\xi^2}\, ds\; e^{ix\xi}\, d\xi$$

$$= \frac{1}{\pi}\int_0^\infty t e^{-st^2} \int_{-\infty}^{\infty} e^{-s\xi^2} e^{ix\xi}\, d\xi\, ds$$

$$= \frac{1}{\pi}\int_0^\infty t e^{-st^2} \int_{-\infty}^{\infty} e^{-s\xi^2} e^{-ix\xi}\, d\xi\, ds \qquad (\text{replace } \xi \text{ by } -\xi)$$

$$= \frac{1}{\pi}\int_0^\infty t e^{-st^2} \left(\frac{\pi}{s}\right)^{1/2} e^{-\frac{x^2}{4s}}\, ds$$

$$= \int_0^\infty \frac{t}{(\pi s)^{1/2}}\, e^{-st^2} e^{-\frac{x^2}{4s}}\, ds$$

The identity is independent of the dimension because only $|x|$ appears which is a scalar. It follows that we have for $x \in \mathbb{R}^n$

$$e^{-t|x|} = \int_0^\infty \left(\frac{t}{(\pi s)^{1/2}}\, e^{-st^2} \right) e^{-\frac{|x|^2}{4s}}\, ds$$

So, $\mathcal{F}(e^{-t|x|}) = \int_0^\infty \frac{t}{(\pi s)^{1/2}} e^{-st^2} \mathcal{F}(e^{-\frac{|x|^2}{4s}})\, ds$

$$= \int_0^\infty \frac{t}{(\pi s)^{1/2}} e^{-st^2} (4\pi s)^{n/2} e^{-s|\xi|^2}\, ds$$

Substituting $k = s(t^2+|\xi|^2)$, we get

$$\mathcal{F}(e^{-t|x|}) = t \int_0^\infty \frac{2^n k^{n/2} \pi^{n/2-1/2}}{(t^2+|\xi|^2)^{n/2}} e^{-k} \frac{dk}{(t^2+|\xi|^2)} \frac{(t^2+|\xi|^2)^{1/2}}{k^{1/2}}$$

$$= 2^n \pi^{\frac{(n-1)}{2}} \frac{t}{(t^2+|\xi|^2)^{(n+1)/2}} \int_0^\infty k^{\frac{n-1}{2}} e^{-k}\, dk$$

$$= 2^n \pi^{\frac{(n-1)}{2}} \frac{t}{(t^2+|\xi|^2)^{(n+1)/2}} \Gamma\left(\frac{n+1}{2}\right)$$

Since $\mathcal{F}^{-1} f = (2\pi)^{-n} (\widetilde{\mathcal{F} f})$, we get

$$\mathcal{F}^{-1}(e^{-t|\xi|}) = \pi^{-(n+1)/2} \frac{t}{(t^2+|x|^2)^{\frac{n+1}{2}}} \Gamma\left(\frac{n+1}{2}\right)$$

Since $e^{-t|x|}$ decays at ∞, the Fourier Transform is smooth. The lack of smoothness of $e^{-t|x|}$ at $x=0$ is reflected as the FT is just having polynomial decay at ∞.

Example 5: Let $f(x) = |x|^{\alpha}$ where $-n < \alpha < 0$

This is not int'ble. But it is locally int'ble.

Also $\displaystyle\int_{|x| \le A} |x|^{\alpha}\, dx = \int_0^A r^{n-1+\alpha} \int_0^{2\pi} d\theta_{n-1} \int_0^{2\pi} d\theta_{n-2} \cdots \int_0^{2\pi} d\theta_1 = C A^{n-1+\alpha}$

So, f is a tempered distribution

We have $\displaystyle\int_0^{\infty} s^{-\alpha/2-1} e^{-s|x|^2}\, ds = \int_0^{\infty} t^{-\alpha/2-1} |x|^{-\alpha-\frac{1}{2}} e^{-t} \frac{dt}{|x|^2}$

(with $t = s|x|^2$)

$$= |x|^{\alpha} \int_0^{\infty} t^{-\frac{\alpha}{2}-1} e^{-t}\, dt$$

$$= |x|^{\alpha}\, \Gamma(-\alpha/2)$$

$$\Rightarrow |x|^{\alpha} = \frac{1}{\Gamma(-\alpha/2)} \int_0^{\infty} s^{-\alpha/2-1} e^{-s|x|^2}\, ds$$

$$\Rightarrow F(|x|^{\alpha}) = \frac{1}{\Gamma(-\alpha/2)} \int_0^{\infty} s^{-\alpha/2-1} F\left(e^{-s|x|^2}\right) ds$$

$$= \frac{1}{\Gamma(-\alpha/2)} \int_0^{\infty} s^{-\alpha/2-1} \left(\frac{\pi}{s}\right)^{n/2} e^{-\frac{|\xi|^2}{4s}}\, ds$$

$$= \frac{\pi^{n/2}}{\Gamma(-\alpha/2)} \int_0^{\infty} s^{-\alpha/2-n/2-1} e^{-\frac{|\xi|^2}{4s}}\, ds$$

Substituting $t = |\xi|^2/4s_0$, we get $s = |\xi|^2/4t$

$$\mathcal{F}(|x|^\alpha)(\xi) = -\frac{\pi^{n/2}}{\Gamma(-\alpha/2)} \int_\infty^0 \left(\frac{|\xi|^2}{4t}\right)^{-\frac{\alpha}{2}-\frac{n}{2}-1} e^{-t} \frac{|\xi|^2}{4t^2}\, dt$$

$$= \frac{\pi^{n/2}}{\Gamma(-\alpha/2)} |\xi|^{-\alpha-n} 2^{\alpha+n} \int_0^\infty t^{\frac{\alpha}{2}+\frac{n}{2}-1} e^{-t}\, dt$$

$$= \frac{\pi^{n/2}\, 2^{\alpha+n}}{\Gamma(-\alpha/2)} \Gamma\left(\frac{\alpha}{2}+\frac{n}{2}\right) |\xi|^{-n-\alpha}$$

Note that $\Gamma(1/2) = \sqrt{\pi}$, $\Gamma(n+1) = n!$ for $n \in \mathbb{Z}_{\geq 0}$.

For $n = 3$, $\alpha = -1$. Then

$$\mathcal{F}(|x|^{-1})(\xi) = \frac{\pi^{3/2}}{\pi^{1/2}} 2^2 |\xi|^{-2} = 4\pi |\xi|^{-2}$$

So, $\mathcal{F}^{-1}(|\xi|^{-2})(x) = \dfrac{1}{4\pi|x|}$

Convolution with tempered distribution

Fix $\psi \in S(\mathbb{R}^n)$, we define convolution of ψ as an operation.
Now, for $\varphi_1, \varphi_2 \in S(\mathbb{R}^n)$

$$\begin{aligned}
\int (\psi * \varphi_1)(x)\, \varphi_2(x)\, dx &= \int\int \psi(x-y)\, \varphi_1(y)\, dy\, \varphi_2(x)\, dx \\
&= \int \varphi_1(y) \int \psi(x-y)\, \varphi_2(x)\, dx\, dy \\
&= \int \varphi_1(y) \int \tilde{\psi}(y-x)\, \varphi_2(x)\, dx\, dy \\
&= \int \varphi_1(y)\, (\tilde{\psi} * \varphi_2)(y)\, dy
\end{aligned}$$

where $\tilde{\psi}(x) = \psi(-x)$

Using the adjoint identity, we define the operation
(& hence the convolution of distribution) as

$$\langle (\psi * f), \varphi \rangle = \langle f, \tilde{\psi} * \varphi \rangle \quad \text{for } \psi, \varphi \in S(\mathbb{R}^n) \ \& \ f \in S'(\mathbb{R}^n)$$

Pp^n : $F(\psi * f) = \hat{\psi} \cdot \hat{f}$ for $\psi \in S(\mathbb{R}^n)$, $f \in S'(\mathbb{R}^n)$

Pf : $\langle F(\psi * f), \varphi \rangle = \langle \psi * f, \hat{\varphi} \rangle = \langle f, \tilde{\psi} * \hat{\varphi} \rangle$

$= \langle f, F(F^{-1}(\tilde{\psi} * \hat{\varphi})) \rangle$

$= \langle \hat{f}, F^{-1}(F(F^{-1}(\tilde{\psi} * \hat{\varphi}))) \rangle$

$= \langle \hat{f}, (2\pi)^n (F^{-1}\tilde{\psi}) \varphi \rangle$ $\left(\because F(h \cdot k) = \dfrac{\hat{h} * \hat{k}}{(2\pi)^n} \right)$

$\left(\because (F^{-1}\tilde{\psi})(x) = \dfrac{1}{(2\pi)^n} \int \psi(-\xi) e^{ix\xi} d\xi = \dfrac{1}{(2\pi)^n} \int \psi(\xi) e^{-ix\xi} d\xi = \dfrac{\hat{\psi}(x)}{(2\pi)^n} \right)$

$= \langle \hat{\psi} \cdot \hat{f}, \varphi \rangle$

Another way to define convolution

for $\psi, \varphi \in S(\mathbb{R}^n)$, $\quad \psi * \varphi = \int \psi(x-y)\,\varphi(y)\,dy$

$$= \int \tau_{-x}\,\tilde{\psi}(y)\,\varphi(y)\,dy$$

Motivated by this, we define

$$(\psi * f)(x) = \langle f, \tau_{-x}\tilde{\psi} \rangle \quad \text{for } f \in S'(\mathbb{R}^n),\ \psi \in S(\mathbb{R}^n)$$

Note that this def^n makes $\psi * f$ a fx^n

Further $\quad P\left(\frac{\partial}{\partial x}\right)(\psi * f)(x) = P\left(\frac{\partial}{\partial x}\right)\langle f, \tau_{-x}\tilde{\psi} \rangle$

$$= \langle f, P\left(\frac{\partial}{\partial x}\right)\tau_{-x}\tilde{\psi} \rangle$$

So, $\psi * f$ is a C^∞-fx^n.

<u>Claim</u> : The distribution $\psi * f$ is tempered & agrees with the previous def^n.

<u>Pf</u> : Let $g(x) = \langle f, \tau_{-x}\tilde{\psi} \rangle$.

Then for $\varphi \in S(\mathbb{R}^n)$, $\quad \int g(x)\varphi(x)\,dx = \int \langle f, \tau_{-x}\tilde{\psi} \rangle\, \varphi(x)\,dx$

$$= \langle f, \int (\tau_{-x}\tilde{\psi})\,\varphi(x)\,dx$$

$$= \langle f, \int \tilde{\psi}(\cdot - x)\,\varphi(x)\,dx \rangle$$

$$= \langle f, \tilde{\psi} * \varphi \rangle$$

Convolution is a smoothening process.

<u>Example</u>: For $\psi \in S(\mathbb{R})$

$$(\psi * \delta)(x) = \langle \delta, \tau_{-x}\tilde{\psi} \rangle = \psi(x-y)\Big|_{x=0} = \psi(x)$$

So, $\psi * \delta = \psi$

This can be verified in another way:

$$F(\psi * \delta) = \hat{\psi} \cdot \hat{\delta} = \hat{\psi} \quad \Rightarrow \quad \psi * \delta = \psi \quad \text{(by Fourier inversion formula)}$$

$$(\because \hat{\delta} = 1)$$

Observe that, $\frac{\partial}{\partial x_k} \psi(x) = \frac{\partial}{\partial x_k} (\psi * \delta)(x) = \left(\psi * \frac{\partial \delta}{\partial x_k}\right)(x)$

$$\left(\because \text{ for } f \in S'(\mathbb{R}^n),\ \psi \in S(\mathbb{R}^n) \quad \frac{\partial}{\partial x_k}(\psi * f)(x) \leftarrow \frac{\partial}{\partial x_k}(\psi * \varphi_n)(x) \text{ for some } \varphi_n \to f \right)$$

$$\frac{\partial}{\partial x_k} \int f(y)\, \psi(x-y)\, dy = \int f(y) \frac{\partial}{\partial x_k} \psi(x-y)\, dy$$

$$= -\int f(y) \frac{\partial}{\partial y_k} \psi(x-y)\, dy \quad \text{(by Chain Rule)}$$

$$= \int \frac{\partial}{\partial y_k} f(y)\, \tau_{-x}\tilde{\psi}(y)\, dy = \left(\psi * \frac{\partial f}{\partial x_k}\right)(x)$$

This implies differentiation of $\psi \in S(\mathbb{R}^n)$ is convolving it with a derivative of δ-fn.

- For $\varphi \in S(\mathbb{R})$

$$(F^{-1}F\varphi)(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \varphi(y)\, e^{-iy\xi}\, dy\; e^{ix\xi}\, dx$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \varphi(y) \int_{-\infty}^{\infty} e^{i(x-y)\xi}\, d\xi\, dy \qquad \left(\begin{matrix}\text{in distribution}\\ \text{sense}\end{matrix}\right)$$

$$= (\varphi * \delta)(x) = \varphi(x)$$

In this way, Fourier inversion can be interpreted as a convolution eq^n. In a sense, the inverse formula '$\delta(x) = \frac{1}{2\pi}\int e^{ix\xi}\, d\xi$' implies the Fourier inversion formula for distributions

Applications to PDEs

For a given fx^n f, the sol^n of $\Delta u = f$ is <u>not</u> unique.

Because if $\Delta u_1 = f$ & $\Delta u_2 = 0$, then $\Delta(u_1+u_2) = f$.

i.e we can add a harmonic fx^n to the sol^n of $\Delta u = f$ to get another sol^n.

Given $f \in S(\mathbb{R}^n)$, if a tempered distr. P satisfies $\Delta P = \delta$, then

$$\Delta(P * f) = (\Delta P) * f = \delta * f = f$$

(Convolution is commutative $\psi * f = f * \psi$)

i.e $P*f$ is a sol^n of $\Delta u = f$

So, the sol^{n}s of $\Delta u = \delta$ are called <u>fundamental sol^{n}s</u> or <u>potentials</u>.

Recall, $\frac{1}{4\pi}\log(x_1^2+x_2^2)$ is a sol^n of $\Delta P = \delta$ for $n=2$.

When $n=3$ (i.e in $\mathbb{R}^3$), $\Delta P = \delta$ gives

$$F(\Delta P) = F(\delta) \Rightarrow -|\xi|^2\hat{P} = 1 \quad -(*) \quad \left(\because F\left(\frac{\partial P}{\partial x_k}\right) = i\xi_k\hat{P}\right)$$

One value of $\hat{P}$ that satisfies $(*)$ is $\hat{P} = -|\xi|^{-2}$

$$\therefore \quad P(x) = -F^{-1}(|\xi|^{-2}) = \frac{-1}{4\pi|x|}$$

(from example 5 of section 4)

This is called <u>Newtonian potential</u>.

As expected, the solⁿ is <u>not</u> unique & if $\hat{P} = -|\xi|^{-2} + g$ satisfying $-|\xi|^2 \hat{P} = 1$ for some distr. g, then $-|\xi|^2 g = 0$

For example $g = \delta$ satisfies the eqⁿ since

$$\langle -|\xi|^2 \delta, \varphi \rangle = \langle \delta, -|\xi|^2 \varphi \rangle = -|\xi|^2 \varphi(\xi)\Big|_{\xi=0} = 0$$

In this case, $P(x) = F^{-1}(-|\xi|^{-2} + \delta) = \dfrac{-1}{4\pi |x|} + \dfrac{1}{(2\pi)^3}$

Note that all solⁿ's of $-|\xi|^2 g = 0$ have pt. support & such distr. must be of the form $g = \sum_{\text{finite}} a_\alpha \left(\frac{\partial}{\partial x}\right)^\alpha \delta$ using 'structure theorem'.

$$\therefore \; F^{-1}(g) = F^{-1}\left(\sum_{\text{finite}} a_\alpha \left(\frac{\partial}{\partial x}\right)^\alpha \delta\right) = \sum_{\text{finite}} k_\alpha a_\alpha x^\alpha$$

In other words, the corresponding fundamental solⁿ's P are of the form : Newtonian potential + a polynomial.

Harmonic fnⁿ's such as $e^{x_1} \cos x_2$ does <u>not</u> appear here because they are <u>not</u> tempered distr.

Note that fn's like $\frac{\partial^2 \delta}{\partial x_1^2}$ cannot be g

(use $m\delta'' = m''(0)\delta - 2m'(0)\delta' + m(0)\delta''$ where $m = -|\xi|^2$)

Solving DE with b'ry cond^n

<u>Example</u> : The eq^n $\Delta(P * f) = f$ is valid for even more general f.

For example, since P is loc. int'ble in both the prev. situations,

$$(P*f)(x) = \int P(x-y)\, f(y)\, dy$$

would define convolution even for f cts. with cmpt. support

& $\Delta(P*f) = (\Delta P) * f = f$

<u>Dirichlet problem</u> : Let D be a b'nd domain in $\mathbb{R}^2$ or $\mathbb{R}^3$ with smooth b'ry B. Let f be a cts. fn^n with support in D & g be cts. on B. We want to solve $\Delta u = f$ in D with $u = g$ on B.

We extend f by setting it equal to 0 outside D & call it F.

So, $\Delta(P * f) = (\Delta P) * f = \delta * F = F = f$ on D.

Set $v = P * f$ restricted to D & so $\Delta v = f$ on D.

Def. $w = u - v$. Then $\Delta w = \Delta u - \Delta v = f - f = 0$ & $w = g - h$ on B where $h = v|_B = P \times f|_B$ $\Big\} \to (*)$

Here h is also cts. So, $(*)$ is the classical Dirichlet problem.
The classical Dirichlet problem has a unique sol^n for w
& for some domain D is given explicit integrals.

Hence $u = v + w$ is the unique sol^n to the original problem.

Example 2: Dirichlet problem for a Half plane
Let D be the half plane for $t > 0$ & $x = (x_1, \dots, x_n) \in \mathbb{R}^n$
We want to solve for fixed $f \in S$,

$$\left(\frac{\partial^2}{\partial t^2} + \Delta x\right) u = 0 \quad \text{in } D \quad \text{with} \quad u(x,0) = f(x)$$

The sol^n is <u>not</u> unique because we can always add ct
(which is harmonic & vanishes at the b'ry)

For uniqueness, we assume that 'u is b'nd'.
We denote by Fx the F.T in x-variable.

Since $u(x,t)$ is b'nd, it is a temporal distr.

We apply F_x to our DE.

$$\frac{\partial^2}{\partial t^2} F_x u(\xi,t) - |\xi|^2 F_x u(\xi,t) = 0 \rightarrow (a)$$

with $F_x u(\xi,0) = \hat{f}(\xi) \rightarrow (b)$

This becomes an ordinary DE. We would be vague by treating distr. $F_x u(\xi,t)$ as fx^n.

On solving a), we get

$$F_x u(\xi,t) = C_1(\xi) e^{t|\xi|} + C_2(\xi) e^{-t|\xi|}$$

Since $e^{t|\xi|}$ is unb'nd, $C_1(\xi) = 0$.

So, $F_x u(\xi, x) = C_2(\xi) e^{-t|\xi|}$

Using (b), we get $\hat{f}(\xi) = F_x u(\xi,0) = C_2(\xi)$

Hence $F_x u(\xi,t) = \hat{f}(\xi) e^{-t|\xi|}$

T'fore, $u(x,t) = F^{-1}(\hat{f}(\xi) e^{-\xi|t|}) = F^{-1}(\hat{f}(\xi) F(F^{-1}(e^{-t|\xi|})))$

$$= f(x) * F^{-1}(e^{-t|\xi|})$$

Recall, $F^{-1}\left(e^{-|\xi|t}\right) = \pi^{-\frac{(n+1)}{2}}\,\Gamma\left(\frac{n+1}{2}\right)\dfrac{t}{(t^2+|x|^2)^{\frac{n+1}{2}}}$ (Poisson kernel of upper half plane)

Hence, $u(x,t) = \pi^{-\frac{(n+1)}{2}}\,\Gamma\left(\frac{n+1}{2}\right)\displaystyle\int f(y)\,\frac{t}{(t^2+|x-y|^2)^{\frac{n+1}{2}}}\,dy$

This is called <u>Poisson integral formula for the upper half plane</u>

- Verify that the sol^n is harmonic (<u>Hint</u>: the kernel is harmonic for $t>0$)

We can conformally transform the Poisson kernel of the disc to the Poisson kernel of the upper half plane for $n=1$.

($\dfrac{z-i}{z+i}$ maps to the upper half plane to the unit disc)

<u>Example</u> : Let $t \geq 0$, $x \in \mathbb{R}^n$ & $u(x,t)$ be b'nd.

The heat eq^n is $\dfrac{\partial u(x,t)}{\partial t} = k \Delta_x u(x,t)$ where k is a +ve const.

We consider t as time, $t=0$ as initial time, x a pt. in space & $u(x,t)$ as temperature. For a fixed $f \in S$, the b'ry cond^n

$$u(x,0) = f(x)$$

is the initial temp.

Applying the partial fourier transform, we get

$$\frac{\partial}{\partial t} F_x u(\xi,t) = -k|\xi|^2 F_x u(\xi,t)$$ with the initial cond^n

$F_x u(\xi,0) = \hat{f}(\xi)$

Solving the DE, we get $F_x u(\xi,t) = c(\xi) e^{-k|\xi|^2 t}$

Using initial cond^n, we obtain $c(\xi) = \hat{f}(\xi)$

So, $F_x u(\xi,t) = \hat{f}(\xi) e^{-k|\xi|^2 t}$

Note that the sol^n is time irreversible.

Hence, $u(x,t) = F^{-1}(\hat{f}(\xi) e^{-k|\xi|^2 t})$

$$= f * F^{-1}(e^{-k|\xi|^2 t})$$

$$= f * \frac{1}{(2\pi)^n} \left(\frac{\pi}{kt}\right)^{n/2} e^{-\frac{|x|^2}{4kt}}$$

$$= \frac{1}{(4\pi kt)^{n/2}} \int f(y)\, e^{-\frac{|x-y|^2}{4kt}}\, dy$$

<u>Heat kernel</u> : $\frac{1}{(4\pi kt)^{n/2}} e^{\frac{-|x|^2}{4kt}}$

<u>B'nd region</u> : $n=1$, $0 \leq x \leq 1$

We consider it with periodic b'ry cond^n.

$$u(0,t) = u(1,t) \quad \text{for all } t.$$

So, $f(0) = f(1)$

We extend f to the whole line as a periodic fn^n of x,

$$f(x+1) = f(x) \quad \forall\, x \in \mathbb{R}$$

So, $u(x,t) = \frac{1}{(4\pi kt)^{1/2}} \int_{-\infty}^{\infty} f(y)\, e^{\frac{-(x-y)^2}{4kt}}\, dy$

$$= \frac{1}{(4\pi kt)^{1/2}} \sum_{j=-\infty}^{\infty} \int_{j}^{j+1} f(y)\, e^{\frac{-(x-y)^2}{4kt}}\, dy$$

By subsituting y by $y-j$ & using $f(y) = f(y-j)$

$$= \frac{1}{(4\pi kt)^{1/2}} \sum_{j=-\infty}^{\infty} \int_0^1 f(y)\, e^{-\frac{(x+j-y)^2}{4kt}}\, dy$$

$$= \int_0^1 \left(\frac{1}{(4\pi kt)^{1/2}} \sum_{j=-\infty}^{\infty} e^{-\frac{(x+j-y)^2}{4kt}} \right) f(y)\, dy$$

This u satisfies $u(x+1, t) = u(x, t)$ for all x & $t \geqslant 0$ i.e

u is periodic in x.

(This can be checked by change of summation variable from

j to $j-1$)

<u>Example 4</u> : Wave eq^n

Consider $\dfrac{\partial^2}{\partial t^2} u(x,t) = k^2 \Delta_x u(x,t)$, $t \in \mathbb{R}$, $x \in \mathbb{R}^n$.

The constant k is 'the maximum propagation speed'

The initial cond^{n}s are $u(x,0) = f(x)$, $\dfrac{\partial}{\partial t} u(x,0) = g(x)$ (Cauchy data)

where $f, g \in S$. This determines a unique sol^n.

Applying Fourier transform, we get

$$\frac{\partial}{\partial t^2} F_x u(\xi,t) = -k|\xi|^2 F_x u(\xi,t)$$

$$F_x u(\xi,0) = \hat{f}(\xi), \quad \frac{\partial}{\partial t} F_x u(\xi,0) = \hat{g}(\xi)$$

Solving the DE, we get

$$F_x u(\xi,t) = c_1(\xi)\cos(k|\xi|t) + c_2(\xi)\sin(k|\xi|t)$$

$$\hat{f}(\xi) = F_x u(\xi,0) = c_1(\xi)$$

$$\hat{g}(\xi) = \frac{\partial}{\partial t} F_x u(\xi,0) = k|\xi|(-c_1(\xi)\sin(k|\xi|t) + c_2(\xi)\cos(k|\xi|t))$$

$$= k|\xi| c_2(\xi)$$

So, $c_2(\xi) = \hat{g}(\xi)/k|\xi|$

So, $\mathcal{F}_x u(\xi,t) = \hat{f}(\xi)\cos(k|\xi|t) + \dfrac{\hat{g}(\xi)}{k|\xi|}\sin(k|\xi|t)$

This is time reversible

We would do the $n=1$ case.

$$\mathcal{F}^{-1}\left(\hat{f}(\xi)\cos(k|\xi|t)\right) = \mathcal{F}^{-1}\left(\hat{f}(\xi)\left(\frac{e^{ik\xi t}+e^{-ik\xi t}}{2}\right)\right)$$

$$= \frac{1}{2}\left(f(x+kt)+f(x-kt)\right)$$

$$\mathcal{F}^{-1}\left(\hat{g}(\xi)\frac{\sin(k|\xi|t)}{k\xi}\right) = \mathcal{F}^{-1}\left(\hat{g}(\xi)\left(\frac{e^{ik\xi t}-e^{-ik\xi t}}{2ik\xi}\right)\right)$$

$$= \mathcal{F}^{-1}\left(\frac{\hat{g}(\xi)}{i\xi}\left(\frac{e^{ik\xi t}-e^{-ik\xi t}}{2k}\right)\right)$$

Let $G(x) = \int_{-\infty}^{x} g(s)\,ds$. By FTC, $\frac{d}{dx}G(x) = g(x)$

So, $i\xi\hat{G}(\xi) = \hat{g}(\xi)$ i.e. $\hat{G}(\xi) = \dfrac{\hat{g}(\xi)}{i\xi}$ for $\xi \neq 0$.

$$= \mathcal{F}^{-1}\left(\hat{G}(\xi)\left(\frac{e^{ik\xi t}-e^{-ik\xi t}}{2k}\right)\right)$$

$$= \frac{1}{2k}\left(G(x+kt)-G(x-kt)\right)$$

$$= \frac{1}{2k}\left(\int_{-\infty}^{x+kt} g(s)\,ds - \int_{-\infty}^{x-kt} g(s)\,ds\right)$$

$$= \frac{1}{2k}\int_{x-kt}^{x+kt} g(s)\,ds$$

Hence, $u(x,t) = \frac{1}{2}\left(f(x+kt) + f(x-kt)\right) + \frac{1}{2k}\int_{x-kt}^{x+kt} g(s)\,ds$

Example 5: Free Schrodinger eq^n

The quantum theory of a single particle is described by complex-valued 'wave function' $\varphi(x)$ defined on $\mathbb{R}^3$.

Here $\int_{\mathbb{R}^3} |\varphi(x)|^2 \, dx$ is assumed to be finite.

Since φ & any multiple of φ describe the same physical state, we assume $\int |\varphi(x)|^2 \, dx = 1$.

The free Schrodinger eq^n is

$$\frac{\partial}{\partial t} u(x,t) = ik \Delta_x u(x,t) \quad \text{where } k > 0.$$

with the initial cond^n $u(x,0) = \varphi(x)$, $\varphi \in S(\mathbb{R}^3)$

Applying F_x, we get

$$\frac{\partial}{\partial t} F_x u(x,t) = -ik|\xi|^2 u(\xi,t)$$

$$F_x u(\xi,0) = \hat{\varphi}(\xi)$$

Solving the DE, we obtain

$$F_x u(\xi,t) = c(\xi) e^{-ik|\xi|^2 t}$$

Using the initial cond^n, we get $F_x u(\xi,t) = \hat{\varphi}(\xi)\, e^{-ik|\xi|^2 t}$

Recall example 3 of section 4 that

$$F^{-1}\left(e^{-ik|\xi|^2 t}\right) = \frac{1}{(2\pi)^3}\left(\frac{\pi}{k|t|}\right)^{3/2} e^{\pm\frac{3\pi i}{4}}\, e^{\frac{i|x|^2}{4kt}}$$

So, the sol^n is $u(x,t) = \dfrac{1}{(4\pi k|t|)^{3/2}}\, e^{\pm\frac{3\pi i}{4}} \displaystyle\int_{\mathbb{R}^3} \varphi(y)\, e^{\frac{i|x-y|^2}{4kt}}\, dy$

where the $\pm$ sign is the sign of t.

The factor $e^{\pm\frac{3\pi i}{4}}$ has no physical significance by the prev. remark.

Since $|F_x u(\xi,t)| = |\hat{\varphi}(\xi)|$, we observe that

$$\int_{\mathbb{R}^3} |u(x,t)|^2\, dx = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} |F_x u(\xi,t)|^2\, d\xi \quad \text{(by Plancherel formula)}$$

is independent of time t & remain normalized.