

MA 5108 Lie Groups and Lie Algebras

Homework 1

1. Let a be an irrational real number and let G be the following subgroup of $GL(2; \mathbb{C})$:

$$G = \left\{ \begin{pmatrix} e^{it} & 0 \\ 0 & e^{ita} \end{pmatrix} : t \in \mathbb{R} \right\}.$$

Show that the closure \bar{G} inside $M_n(\mathbb{C})$ satisfies:

$$\bar{G} = \left\{ \begin{pmatrix} e^{it} & 0 \\ 0 & e^{is} \end{pmatrix} : t, s \in \mathbb{R} \right\}.$$

You may use without proof that the set of numbers of the form $e^{2\pi i n a}$, $n \in \mathbb{Z}$, is dense in S^1 .

2. Describe a natural 1-to-1 correspondence between elements of $SO(3)$ and elements of

$$T^1 S^2 = \{(p, v) \in \mathbb{R}^3 \times \mathbb{R}^3 : \|p\| = \|v\| = 1 \text{ and } p \perp v\},$$

which can be thought of as the collection of all unit vectors v tangent to all points p of S^2 .

3. Consider the symmetric bilinear form $[\cdot, \cdot]_{n,k}$ on \mathbb{R}^{n+k} defined by the formula

$$[x, y]_{n,k} = x_1 y_1 + \dots + x_n y_n - x_{n+1} y_{n+1} - \dots - x_{n+k} y_{n+k}$$

The set of $(n+k) \times (n+k)$ real matrices A which preserve this form, i.e., such that

$$[Ax, Ay]_{n,k} = [x, y]_{n,k} \text{ for all } x, y \in \mathbb{R}^{n+k}$$

is called *the generalized orthogonal group* $O(n; k)$. Let g be the $(n+k) \times (n+k)$ diagonal matrix given by

$$g = \begin{pmatrix} I_n & 0 \\ 0 & -I_k \end{pmatrix}.$$

- (a) Show that for all $x, y \in \mathbb{R}^{n+k}$, we have $[x, y]_{n,k} = \langle x, gy \rangle$.
- (b) Show that an $(n+k) \times (n+k)$ real matrix A is in $O(n; k)$ if and only if $A^T g A = g$.
- (c) Show that $O(n; k)$ is a subgroup of $GL(n+k; \mathbb{R})$ and is a matrix Lie group.

4. Define the real and imaginary parts of a quaternion as follows:

$$\begin{aligned} \operatorname{Re}(a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}) &= a \\ \operatorname{Im}(a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}) &= b\mathbf{i} + c\mathbf{j} + d\mathbf{k}. \end{aligned}$$

Let $q_1 = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}$ and $q_2 = x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}$ be purely imaginary quaternions in \mathbb{H} . Prove that $-\operatorname{Re}(q_1 \cdot q_2)$ is their vector dot product in $\mathbb{R}^3 = \operatorname{span}\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ and $\operatorname{Im}(q_1 \cdot q_2)$ is their vector cross product.

5. In this exercise we aim to show that $\mathrm{GL}(n; \mathbb{C})$ is isomorphic to a subgroup of $\mathrm{GL}(2n; \mathbb{R})$. In general, for $A \in M_n(\mathbb{K})$, where $\mathbb{K} = \mathbb{R}$ or \mathbb{C} , we will denote by T_A the linear transformation

$$T_A : \mathbb{K}^n \rightarrow \mathbb{K}^n, \quad v \mapsto A \cdot v.$$

We will denote by f_n the bijection

$$f_n : \mathbb{C}^n \rightarrow \mathbb{R}^{2n}, \quad (x_1 + iy_1, x_2 + iy_2, \dots, x_n + iy_n) \mapsto (x_1, y_1, x_2, y_2, \dots, x_n, y_n).$$

Consider the function $\rho_1 : M_1(\mathbb{C}) \rightarrow M_2(\mathbb{R})$, $a + bi \mapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$.

Define $\rho_n : M_n(\mathbb{C}) \rightarrow M_{2n}(\mathbb{R})$ so that for $A \in M_n(\mathbb{C})$, $\rho_n(A) \in M_{2n}(\mathbb{R})$ is the $2n \times 2n$ matrix obtained by replacing the (i, j) th entry a_{ij} by the 2×2 block $\rho_1(a_{ij})$.

- (a) Prove that for each $A \in M_n(\mathbb{C})$ we have $f_n \circ T_A = T_{\rho_n(A)} \circ f_n$.
- (b) Prove that for all $A, B \in M_n(\mathbb{C})$, we have $\rho_n(A \cdot B) = \rho_n(A) \cdot \rho_n(B)$.
- (c) Prove that ρ_n is injective but not surjective.

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1. Determine the center $Z(H)$ of the Heisenberg group H . Show that the quotient group $H/Z(H)$ is abelian.
2. Show that $SO(n)$ is path-connected. (See [Hall, Exercise 13] for a hint.)
3. Using the polar decomposition of $SL(n; \mathbb{R})$ and the path-connectedness of $SO(n)$ (previous exercise), show that $SL(n; \mathbb{R})$ is path-connected.
4. Suppose $G \subset GL(n_1; \mathbb{C})$ and $H \subset GL(n_2; \mathbb{C})$ are matrix Lie groups and $\Phi : G \rightarrow H$ is a Lie group homomorphism. Is $\Phi(G)$ necessarily a matrix Lie group? Prove or give a counter-example.
5. Let $f : SU(n) \times U(1) \rightarrow U(n)$ be given by $f(A, (\lambda)) :=$ the result of multiplying each entry of the first row of A by λ . Prove that f is a homeomorphism but not a homomorphism.

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Homework 3

1. Suppose P is a real, positive, symmetric matrix with determinant one. Show that there is a unique real, positive, symmetric matrix Q whose square is P .

Hint: Consider two real, positive, symmetric square roots Q_1 and Q_2 of P and show that the eigenspaces of both Q_1 and Q_2 coincide with the eigenspaces of P .

2. Determine the Lie algebra \mathfrak{h} of the Heisenberg group H .
3. Complete the proof that the Lie algebra of $U(n)$ is $\mathfrak{u}(n)$: describe a natural basis of $\mathfrak{u}(n)$ and construct a path in $U(n)$ tangent to each element of the basis.
4. Prove that the Lie algebra of $\rho_n(\mathrm{GL}(n; \mathbb{C})) \subset \mathrm{GL}(2n; \mathbb{R})$ is equal to $\rho_n(\mathfrak{gl}(n; \mathbb{C}))$.

5. Prove that the tangent space to a matrix Lie group G at any $A \in G$ is:

$$T_A(G) = \{BA : B \in \mathfrak{g}\} = \{AB : B \in \mathfrak{g}\}.$$

Here $\mathfrak{g} = T_I(G)$.

6. Suppose $\{A_\ell\}$ and $\{B_\ell\}$ are sequences of matrices in $M_n(\mathbb{K})$, where $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Suppose that $\sum A_\ell$ converges absolutely, and that $\sum A_\ell = A$ and $\sum B_\ell = B$. Let $C_\ell := \sum_{k=0}^{\ell} A_k B_{\ell-k}$. Prove that $\sum C_\ell = A \cdot B$.

7. Let $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Calculate e^A .

8. Describe all one parameter groups in $\mathrm{GL}(1; \mathbb{C})$. Draw some in the xy -plane.

9. Let $A = \begin{bmatrix} p & q \\ -q & p \end{bmatrix} \in \mathfrak{gl}(2; \mathbb{R})$.

(a) Calculate e^A .

(b) Draw the vector field T_A when $p = 1$ and $q = 2$, and sketch some integral curves.

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Homework 4

1. Let $\{Y_k\}$ be a sequence of points in \mathbb{R}^m such that as $k \rightarrow \infty$, $\|Y_k\|$ converges to 0 and $\frac{Y_k}{\|Y_k\|}$ converges to Y . Let $t \in \mathbb{R}$. Prove that there exists a sequence of integers N_k such that $N_k Y_k$ converges to tY .

2. In class we proved that

(i) There exists $r > 0$ such that $V := \exp(B_r)$ is a neighbourhood of I in $\mathrm{GL}(n; \mathbb{K})$ and $\exp : B_r \rightarrow V$ is a homeomorphism.

(ii) If $G \subset \mathrm{GL}(n; \mathbb{K})$ is a matrix Lie group with Lie algebra \mathfrak{g} then in part (i), r can be chosen such that additionally $\exp(B_r \cap \mathfrak{g}) = \exp(B_r) \cap \mathfrak{g}$.

Give an example to show that for part (ii) it may be necessary to choose a strictly smaller r than the one given in part (i).

Hint: Define G similar to the one in HW 1.1, except with a now being a suitably chosen rational number.

3. Prove that $S^n \subset \mathbb{R}^{n+1}$ is an n -dimensional manifold.

4. Prove that the cone $\{(x, y, z) \in \mathbb{R}^3 : z = \sqrt{x^2 + y^2}\} \subset \mathbb{R}^3$ is not a manifold.

5. If $X_1 \subset \mathbb{R}^{m_1}$ and $X_2 \subset \mathbb{R}^{m_2}$ are manifolds whose dimensions are d_1 and d_2 , prove that

$$X_1 \times X_2 = \{(p_1, p_2) \in \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \cong \mathbb{R}^{m_1+m_2} : p_1 \in X_1, p_2 \in X_2\}$$

is a manifold of dimension $d_1 + d_2$.

6. Let $X \subset \mathbb{R}^m$ be an n -dimensional manifold. Define the *tangent bundle* of X as:

$$TX := \{(p, v) \in \mathbb{R}^m \times \mathbb{R}^m \cong \mathbb{R}^{2m} : p \in X \text{ and } v \in T_p X\}.$$

Prove that TX is a manifold of dimension $2n$.

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Homework 6

Note: If $G_1 \subset GL(n_1; \mathbb{C})$ and $G_2 \subset GL(n_2; \mathbb{C})$ are matrix Lie groups, $G_1 \times G_2$ can be thought of as a matrix Lie group by viewing it as a subgroup of $GL(n_1+n_2; \mathbb{C})$, via the block diagonal inclusion of $GL(n_1; \mathbb{C}) \times GL(n_2; \mathbb{C})$ into $GL(n_1+n_2; \mathbb{C})$.

1. In this exercise we want to prove the existence of a surjective map $F : SU(2) \times SU(2) \rightarrow SO(4)$ which is a 2-to-1 Lie group homomorphism and a local diffeomorphism.

(a) Consider the vector space $V := \left\{ \begin{bmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{bmatrix} \mid z_1, z_2 \in \mathbb{C} \right\} \cong \mathbb{R}^4$. Show that for each $(g_1, g_2) \in SU(2) \times SU(2)$, there is an invertible linear map $F_{g_1, g_2} : V \rightarrow V$ given by $v \mapsto g_1 v g_2^*$.

(b) Let $\mathcal{B} := \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\}$; this is a basis for V . Show that

$$F : SU(2) \times SU(2) \rightarrow GL(4; \mathbb{R}), \quad (g_1, g_2) \mapsto [F_{g_1, g_2}]_{\mathcal{B}}$$

defines a Lie group homomorphism, whose image lies in $SO(4)$.

(c) By part (b), we actually have a map $F : SU(2) \times SU(2) \rightarrow SO(4)$. Determine $\ker(F)$.

(d) Show that $DF_{(I, I)} : \mathfrak{su}(2) \times \mathfrak{su}(2) \rightarrow \mathfrak{so}(4)$ is invertible.

Hint: Determine the map $DF_{(I, I)}$ on basis vectors. It may be useful to work with the

basis for $\mathfrak{su}(2)$ given by $\{X_1, X_2, X_3\} := \left\{ \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\}$, and the

basis for $\mathfrak{su}(2) \times \mathfrak{su}(2)$ given by $\{(X_1, 0), (X_2, 0), (X_3, 0), (0, X_1), (0, X_2), (0, X_3)\}$.

(e) Prove that the map $F : SU(2) \times SU(2) \rightarrow SO(4)$ obtained by parts (a)-(c) above, is surjective, and is a local diffeomorphism.

Note: The center $\mathfrak{z}(\mathfrak{g})$ of a Lie algebra \mathfrak{g} is defined as $\mathfrak{z}(\mathfrak{g}) = \{X \in \mathfrak{g} \mid [X, Y] = 0 \text{ for all } Y \in \mathfrak{g}\}$.

2. Let H be the Heisenberg group and \mathfrak{h} its Lie algebra.

(a) Determine $\mathfrak{z}(\mathfrak{h})$.

(b) For any $X, Y \in \mathfrak{h}$, show that $[X, Y] \in \mathfrak{z}(\mathfrak{h})$.

3. Prove that $\exp : \mathfrak{sl}(2; \mathbb{C}) \rightarrow SL(2; \mathbb{C})$ is not onto.

4. Give an example of matrices X and Y in $\mathfrak{sl}(2; \mathbb{C})$ such that there does not exist any Z in $\mathfrak{sl}(2; \mathbb{C})$ with $e^X e^Y = e^Z$.

MA 5108 Lie Groups and Lie Algebras Homework 5

1. Give two proofs, one working with the matrix Lie groups and one working with the Lie algebras, that $\mathrm{SO}(n)$ is not abelian.
2. Give two proofs, one working with the matrix Lie groups and one working with the Lie algebras, that $\mathrm{SO}(3)$ and the Heisenberg group H are not isomorphic.
3. Let G_1, G_2 be matrix Lie groups with Lie algebras $\mathfrak{g}_1, \mathfrak{g}_2$. Let $f : G_1 \rightarrow G_2$ be a Lie group homomorphism. If $Df_1 : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ is bijective, prove that $Df_g : T_g G_1 \rightarrow T_g G_2$ is bijective for all $g \in G_1$.
4. Prove that:
 - (a) Up to Lie algebra isomorphism, there is exactly one one-dimensional real Lie algebra.
 - (b) Up to Lie algebra isomorphism, there are exactly two two-dimensional real Lie algebras.

Note: If \mathfrak{g} is a Lie algebra, then a subalgebra \mathfrak{h} is called an ideal if $[X, H] \in \mathfrak{h}$ for all $X \in \mathfrak{g}$ and $H \in \mathfrak{h}$.

5. If \mathfrak{g} is a Lie algebra, the center of \mathfrak{g} is the set of all $Z \in \mathfrak{g}$ such that $[X, Z] = 0$ for all $X \in \mathfrak{g}$. Prove that the center of \mathfrak{g} is an ideal.
6. If $\phi : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ is a Lie algebra homomorphism, show that $\ker(\phi)$ is an ideal in \mathfrak{g}_1 .

7. Let G be a matrix Lie group. Prove that $\mathrm{Ad} : G \rightarrow \mathrm{GL}(\mathfrak{g})$ is continuous.