MA5108

Lie Groups & Lie Algebras

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Grading Policy :

Midsem	30%
Endsem	40%
In-semester (Weekly HW)	30%

Reference : Stilwell

Hall

Tapp

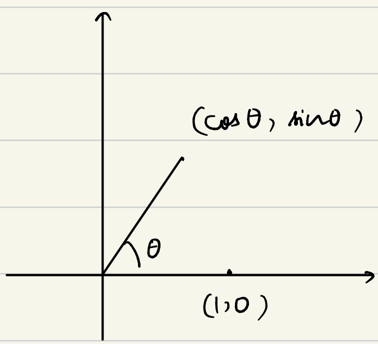
Introduction & Motivation

- Discrete gps.: $\mathbb{Z}_n$, D_n
  describe symmetry gps. of geometrical objects

- Lie gps. arise as gps. of 'cont. symm'.

eg: Rot^{n}s of the plane $\mathbb{R}^2$.

$$R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$$
$$e_1 \mapsto (\cos\theta, \sin\theta)$$
$$e_2 \mapsto (-\sin\theta, \cos\theta)$$



$$[R_\theta] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Note: $R_\theta \circ R_\varphi = R_{\theta+\varphi}$

The set $\{[R_\theta] : \theta \in \mathbb{R}\}$ is a gp. under matrix multip^n.
It is called the <u>special orthogonal gp.</u> $SO(2)$.

Also, $SO(2) = \left\{ \begin{array}{l} 2\times 2 \text{ matrices } A \text{ with real entries s.t} \\ AA^T = Id \ \& \ \det A = 1 \end{array} \right\}$

Geometrically, $SO(2)$ can be identified with the unit circle.

eg: rigid motions (or isometries) of $\mathbb{R}^3$ are maps from $\mathbb{R}^3$ to $\mathbb{R}^3$ that preserve the Euclidean dist. b/w pairs of pts.

the set of all rigid motions of $\mathbb{R}^3$ that
- keep the origin fixed.
- send $B = \{e_1, e_2, e_3\}$ to another positively oriented basis (i.e preserve orientation)

These can be naturally identified with $SO(3)$.

Most generally, a Lie gp. G is a set that has both the structure of a gp. $(G, \cdot)$ & the structure of a differentiable manifold s.t these two structures 'behave well' with each other i.e the maps.

$$p : G \times G \to G$$
$$(a, b) \mapsto a \cdot b$$

$$i : G \to G$$
$$a \mapsto a^{-1}$$

are differentiable

Given a Lie gp G, its tangent space at e, T_eG is a vector sp. with a bilinear map

$$[\cdot, \cdot] : T_eG \times T_eG \to T_eG$$

which satisfies the following $\forall\ X, Y, Z \in T_eG$

- Alternating : $[X, Y] = -[Y, X]$
- Jacobi identity:

$$[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0$$

Such a pair $(T_e G, [\cdot,\cdot])$ is called a Lie Algebra.

eg: $(\mathbb{R}^3, \times)$
└ cross product

In this course, we'll be focusing on Matrix Lie Gps.

The general linear gp. over reals denoted by $GL(n, \mathbb{R})$ or $GL_n(\mathbb{R})$ is the gp. of all $n \times n$ invertible matrices with real entries over matrix multip^n.

Sim, $GL_n(\mathbb{C})$ can be defined.

$M_n(\mathbb{C})$: set of all $n \times n$ matrices with complex entries

Note that $M_n(\mathbb{C})$ can be identified with $\mathbb{C}^{n^2}$.

Let $\{A_n\}$ be a seq. of matrices in $M_n(\mathbb{C})$.
We say that A_m converges to A if $A_m \to A$ in $\mathbb{C}^{n^2}$
i.e if $(A_m)_{kl} \to (A)_{kl}$ as $m \to \infty$

A <u>matrix Lie gp.</u> is any subgp. G of $GL_n(\mathbb{C})$ s.t if $\{A_n\}$ is any seq of matrices in G, & A_n converges to A, then either A is <u>not</u> invertible or $A \in G$

<u>Note</u>: This condⁿ is saying that G is a closed subset of $GL_n(\mathbb{C})$.

So an eq. defⁿ is that matrix Lie gp. is a closed subset of $GL_n(\mathbb{C})$.

<u>eg</u>: - $GL_n(\mathbb{C})$, $GL_n(\mathbb{R})$,

- $GL_n(\mathbb{Q})$ is <u>not</u> a matrix Lie gp.

- Special linear gps. : $SL_n(\mathbb{R})$ & $\underbrace{SL_n(\mathbb{C})}$

gp. of inv. $n \times n$ matrices whose det is 1.

This is a subgp. of $GL_n(\mathbb{C})$.

Also if a seq. of matrices with $\det(A_m) = 1$ & $A_m \to A$, then $\det(A) = 1$ since det is a cont. fxⁿ.

Hence, $SL_n(\mathbb{C})$ is a closed subset of $GL_n(\mathbb{C})$.

– The orthogonal & special orthogonal gps.

$O(n)$ & $SO(n)$.

Recall: The std. inner product on $\mathbb{R}^n$:

$$\langle x, y\rangle = \sum_{k=1}^{n} x_k y_k$$

An $n \times n$ real matrix is orthogonal if it preserves the inner prod.

$$\langle Ax, Ay\rangle = \langle x, y\rangle$$

<u>Ppn</u>: TFAE

1. A is orthogonal
2. the col. vecs. of A form an o.n basis of $\mathbb{R}^n$.
3. $A^T A = Id$

$O(n) :=$ set of all $n \times n$ real orthogonal matrices.

Ppn: $O(n)$ is a matrix Lie gp.

Pf: 1. If $A \in O(n)$, then $\det(A) = \pm 1$
so each orth. matrix is inv.
So, $O(n) \subseteq GL_n(\mathbb{C})$.

2. If $A \in O(n)$, then $A^{-1} \in O(n)$:
$\langle A^{-1}x, A^{-1}y \rangle = \langle AA^{-1}x, AA^{-1}y \rangle = \langle x, y \rangle$

3. If $A, B \in O(n)$, then $AB \in O(n)$:
$\langle ABx, ABy \rangle = \langle Bx, By \rangle = \langle x, y \rangle$

So, $O(n)$ is a subgp. of $GL_n(\mathbb{C})$.

4. If $\{A_n\} \subseteq O(n)$ & $A_m \to A$, then, $A_m A_m^T = Id \ \forall m$
implies $AA^T = Id$.

So, $A \in O(n)$.

$SO(n) :=$ set of all $n \times n$ orth. matrices with det 1.

$SO(n)$ is a subgp. of $O(n)$.
$SO(n)$ is a matrix Lie gp.
($A^TA = I$, $\det A = 1$ are closed cond^{n}s under taking limits)

- The unitary & special unitary gps
  $U(n)$ & $SU(n)$

The std. inner prod. on $\mathbb{C}^n$:

$$\langle x, y \rangle = \sum_{k=1}^{n} x_k \overline{y_k}$$

A complex $n \times n$ matrix A is unitary if

$$\langle Ax, Ay \rangle = \langle x, y \rangle \quad \forall\, x, y \in \mathbb{C}^n$$

<u>Pp^n</u>: TFAE

1. A is unitary
2. the col. vecs. of A form o.n.b of $\mathbb{C}^n$.
3. $A^*A = Id$

Note: $|\det(A)| = 1 \quad \forall$ unitary matrices A.

$U(n) :=$ the set of all $n \times n$ unit. matrices.

$SU(n) :=$ the set of all $n \times n$ unit. matrices with det 1.

$U(n)$ & $SU(n)$ are matrix Lie gps.

Orthogonal matrices & isometries

Euclidean dist. on $\mathbb{R}^n$.

$$d_E(x, y) = \|x - y\| = \sqrt{\langle x-y, x-y \rangle}$$

Isometry: A fn $f : \mathbb{R}^n \to \mathbb{R}^n$ is called an isometry if for all $x, y \in \mathbb{R}^n$,

$$d_E(f(x), f(y)) = d_E(x, y)$$

Pp^n:

1. If $A \in O(n)$, then the linear transformation left-multiplication by A
$A: \mathbb{R}^n \to \mathbb{R}^n$ is an isometry.

2. If $f: \mathbb{R}^n \to \mathbb{R}^n$ is an isometry with $f(0)=0$ then, $f = A$ for some $A \in O(n)$.
In particular f is linear.

Pf: 1. Let $A \in O(n)$ & $x, y \in \mathbb{R}^n$.
Then $d_E(Ax, Ay) = \|Ax - Ay\|$
$= \|x-y\| = d_E(x,y)$

So, A is an isometry.

2. Let $f: \mathbb{R}^n \to \mathbb{R}^n$ be isometry s.t $f(0)=0$.
Let $x \in \mathbb{R}^n$.

$\|f(x)\| = d_E(f(x), 0) = d_E(f(x), f(0)) = d_E(x, 0) = \|x\|$
$\forall x \in \mathbb{R}^n$.

So, f preserves norm.

Using the 'polarization identity'

$$\langle x, y \rangle = \frac{1}{2}\left(\|x-y\|^2 - \|x\|^2 - \|y\|^2\right)$$

We see that f also preserves inner prod.

Let A be the matrix whose ith col. is $f(e_i)$,
then $A \in O(n)$.

Def. $g : \mathbb{R}^n \to \mathbb{R}^n$ by $g = A^{-1} \circ f$
Then g is an isometry & $g(0) = 0$, so g preserves norms & inner prod. & $g(e_i) = e_i \ \forall \ i = 1, \dots, n$.

Let $x \in \mathbb{R}^n$, $\quad x = \sum_{i=1}^{n} a_i e_i$

$$g(x) = \sum_{i=1}^{n} b_i e_i$$

Then $b_i = \langle g(x), e_i \rangle = \langle g(x), g(e_i) \rangle = \langle x, e_i \rangle = a_i$

So, $g(x) = x$

So, g is identity fn^n from $\mathbb{R}^n$ to $\mathbb{R}^n$

Hence $f = A$.

$O(n)$ is the gp. of isometries of $\mathbb{R}^n$ that fix origin

Cor: $O(n)$ maps $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ to itself.

Skew field: A skew-field is a set $\mathbb{K}$ with op. $+$ & $\cdot$ satisfying

1. $a \cdot (b+c) = a \cdot b + a \cdot c$
   $(b+c) \cdot a = b \cdot a + c \cdot a$

2. $\mathbb{K}$ is an abelian gp. under $+$, with id. elem. 0.

3. $\mathbb{K} \setminus \{0\}$ is a gp. under $\cdot$, with id. elem. 1.

A skew-field in which multip^n is commutative, is called a field.

Quaternions

Denote $(a,b,c,d) \in \mathbb{R}^4$ symbolically as : $a+bi+cj+dk$

Define a multip^n rule for the symbols $\{i,j,k\}$ as follows :

- $i \cdot 1 = 1 \cdot i = i$

  $j \cdot 1 = 1 \cdot j = j$

  $k \cdot 1 = 1 \cdot k = k$

- $i^2 = j^2 = k^2 = -1$

- $i \cdot j = k, \quad j \cdot k = i, \quad k \cdot i = j$

  $j \cdot i = -k, \quad k \cdot j = -i, \quad i \cdot k = -j$

This multip^n extends linearly to a multip^n on all of $\mathbb{R}^4$.

$\mathbb{R}^4$ with component-wise add^n & the above multip^n satisfies the cond^{n}s of a skew-field called the set of quaternions & is denoted by $\mathbb{H}$.

We define the conjugate & the norm of any element $q = a + bi + cj + dk \in \mathbb{H}$ as follows:

$$\bar{q} = a - bi - cj - dk$$

$$|q| = (a^2 + b^2 + c^2 + d^2)^{1/2}$$

$$q \cdot \bar{q} = \bar{q} \cdot q = |q|^2$$

Hence, $\dfrac{\bar{q}}{|q|^2}$ is a multiplicative inv. for q.

Finally, also $\overline{q_1 \cdot q_2} = \bar{q_1} \cdot \bar{q_2}$

$|q_1 \cdot q_2| = |q_1|\,|q_2|$

Note: 1. Natural inclusions $\mathbb{R} \subseteq \mathbb{C} \subseteq \mathbb{H}$

2. for $x \in \mathbb{R}$, $q \in \mathbb{H}$, $x \cdot q = q \cdot x$

Hence, $\mathbb{H}$ is a vector sp. over $\mathbb{R}$.

But in general for $q_1, q_2 \in \mathbb{H}$, $q_1 \cdot q_2 \neq q_2 \cdot q_1$

Symplectic group $Sp(n)$

The std. inner prod. on $\mathbb{H}^n$ is :

$$\langle x, y\rangle = \sum_{k=1}^{n} \overline{x_k}\, y_k \qquad \forall\ x, y \in \mathbb{H}^n$$

$$Sp(n) := \{ A \in GL(n, \mathbb{H}) : \langle Ax, Ay\rangle = \langle x, y\rangle\ ,\ \forall\ x, y \in \mathbb{H}^n \}$$

Ppn: for $A \in GL(n, \mathbb{H})$, the following are equiv.

1. $A \in Sp(n)$
2. the cols. of A form an o.n.b of $\mathbb{H}^n$
3. $A^*A = I$

There is a natural way to view $GL(n, \mathbb{H})$ as a subgp. of $GL(2n, \mathbb{C})$ & one can prove that $Sp(n)$ is a matrix Lie gp.

<u>Note</u>: Also, it is possible to view $GL(n, \mathbb{C})$ as a subgp. of $GL(2n, \mathbb{R})$

eg:
$$[e^{i\theta}] \in GL(1, \mathbb{C})$$

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \in GL(2, \mathbb{R})$$

Heisenberg gp. H

$H := \{$ 3x3 real matrices A of the form

$$A = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{R} \}$$

At least, $H \subseteq GL(3, \mathbb{R}) \subseteq GL(3, \mathbb{C})$

It can be checked that H is closed under matrix multip^n.

Also, clearly $I \in H$.

Also, $A^{-1} = \begin{pmatrix} 1 & -a & ac-b \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{pmatrix}$

Hence, H is a subgp. of $GL(3, \mathbb{R})$

H is also closed under limits.

So, H is a matrix Lie gp.

Low-dimensional examples from geometric viewpt.

- $O(1) = \{(1), (-1)\}$

  $SO(1) = \{(1)\}$

- $$\underset{S^1}{\overset{\updownarrow}{SO(2)}} = \left\{ \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \ \theta \in [0, 2\pi) \right\}$$

  $$\underset{S^1 \sqcup S^1}{\overset{\updownarrow}{O(2)}} = SO(2) \cup \left\{ \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}, \ \theta \in [0, 2\pi) \right\}$$

- $$SU(2) = \left\{ A = \begin{bmatrix} z_1 & z_2 \\ z_3 & z_4 \end{bmatrix} : A^*A = Id, \ \det A = 1 \right\}$$

i.e $$\begin{pmatrix} \bar{z}_1 & \bar{z}_3 \\ \bar{z}_2 & \bar{z}_4 \end{pmatrix} = A^* = A^{-1} = \frac{1}{\underbrace{\det A}_{1}} \begin{bmatrix} z_4 & -z_2 \\ z_3 & z_1 \end{bmatrix}$$

$\Rightarrow \quad z_4 = \bar{z}_1 \quad , \quad z_3 = -\bar{z}_2$

$$A = \begin{bmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{bmatrix}, \quad \text{also} \quad \det A = 1 \Rightarrow |z_1|^2 + |z_2|^2 = 1$$

So, $$SU(2) = \left\{ A = \begin{bmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{bmatrix} : |z_1|^2 + |z_2|^2 = 1 \right\}$$

So, by identifying $\mathbb{C}^2 \longleftrightarrow \mathbb{R}^4$

$$(x_1 + iy_1, x_2 + iy_2) \longmapsto (x_1, y_1, x_2, y_2)$$

$SU(2)$ can be thought of as unit 3-dimensional sphere inside $\mathbb{R}^4$.

Hence, $S^3 \subseteq \mathbb{R}^4$ has a gp. structure

$$SU(2) \subseteq GL(2, \mathbb{C}) \subseteq M_2(\mathbb{C}) \simeq \mathbb{C}^4 \simeq \mathbb{R}^8$$

- $U(1) = \{ [e^{i\theta}] \in GL(1, \mathbb{C}) : \theta \in [0, 2\pi) \}$

$$\begin{array}{c} U(1) \\ \updownarrow \\ S^1 \end{array}$$

$SU(1) = \{ [1] \}$

- $Sp(1) = \{ [a+bi+cj+dk : a^2+b^2+c^2+d^2 = 1 \}$

can be identified with set of unit quaternions, thus also identified with the 3-dimensional sphere S^3 in $\mathbb{R}^4$

$$Sp(1) \subseteq \mathbb{H} \simeq \mathbb{R}^4$$

Topological Properties

- **Compactness:**

A matrix Lie gp. G is said to compact if it is compact as a subset of $\underbrace{\mathbb{C}^{n^2}}_{M_n(\mathbb{C})} \simeq \mathbb{R}^{2n^2}$ with Euclidean topology.

Equiv. (by Heine-Borel), G is compact if it is closed & bounded in $M_n(\mathbb{C})$ i.e if it satisfies both of these cond^{n}s:

1. If A_m is any seq. of matrices in G, and A_m conv. to A in topo. of $M_n(\mathbb{C})$, then $A \in G$

2. $\exists$ const. C s.t $\forall A \in G$, $|A_{ij}| \leq C$ $\forall 1 \leq i, j \leq n$

eg: $O(n)$, $SO(n)$, $U(n)$, $SU(n)$, $Sp(n)$ are compact

$GL(n, \mathbb{R})$, $GL(n, \mathbb{C})$, $\underset{n \neq 1}{SL(n, \mathbb{R})}$, $\underset{n \neq 1}{SL(n, \mathbb{C})}$ are not compact.

- <u>Path connectedness</u> :

A matrix Lie gp. is said to be path-connected if for every pair $A, B \in G$, $\exists$ a cont. fxn (called a path) $f : [0,1] \to G$ s.t $f(0) = A$ & $f(1) = B$.

A matrix Lie gp. which is <u>not</u> path-connected can be expressed as the disjoint union of several path-connected pieces, called path-components.

Precisely, define an eq. relⁿ $\sim$ on G by setting $A \sim B$ if $\exists$ a path in G from A to B.

The eq. classes are called path-components.

eg: <u>1.</u> $GL(n, \mathbb{R})$ is <u>not</u> path-connected, since if $\det A > 0$, $\det B < 0$ & $f : [0,1] \to M_n(\mathbb{R})$ with $f(0) = A$ & $f(1) = B$, then $\exists\ s \in (0,1)$ s.t $\det(f(s)) = 0$, i.e $f(s) \in M_n(\mathbb{R}) \setminus GL(n, \mathbb{R})$ so f cannot be completely in $GL(n, \mathbb{R})$.

(set of $\det = 0$ matrices has dimension 1 less than total dim of $M_n(\mathbb{R})$)

<u>2.</u> $GL(n, \mathbb{C})$ is path-connected.

(set of $\det = 0$ matrices has dimension 2 less than total dim of $M_n(\mathbb{C})$)

$\underline{\text{Pp}^n}$: If G is a matrix Lie gp., then the path-comp. of G containing the identity (denoted G_0) is a subgp. of G.

$\underline{\text{Pf}}$: If $A, B \in G_0$, then there are cont. paths $A(t)$, $B(t)$ s.t

$A(0) = A$, $A(1) = I$

$B(0) = B$, $B(1) = I$

Then, $f_1(t) = A(t)B(t)$ is a path in G joining AB to I.
So, $AB \in G_0$.

Also, $f_2(t) = (A(t))^{-1}$ is a path in G joining A^{-1} to I.
So, $A^{-1} \in G_0$

Hence, G_0 is a subgp. of G.

Ppn: $GL(n,\mathbb{C})$ is path-connected $\forall\, n \geq 1$.

Pf: If $n=1$, $GL(n,\mathbb{C}) \simeq \mathbb{C}^* = \mathbb{C}\setminus\{0\}$ is clearly path-conn.

For $n \geq 2$, enough to show that any $A \in GL(n,\mathbb{C})$ can be connected to $I \in GL(n,\mathbb{C})$ by a path in $GL(n,\mathbb{C})$.

Given $A \in GL(n,\mathbb{C})$, $\exists$ invertible C & upper triangular matrix B s.t $A = CBC^{-1}$ i.e $B = \begin{bmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$ each $\lambda_i \neq 0$ ($\because A$ is invertible)

$$B = D + B_1 = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} + \begin{bmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{bmatrix}$$

$A_1(t) = C(D+(1-t)B_1)C^{-1}$ is a path with $A_1(0) = A$ & $A_1(1) = CDC^{-1}$

As in the case $n=1$, $\exists$ path $\lambda_i(t)$ with $\lambda_i(1) = \lambda_i$ & $\lambda_i(2) = 1$

Then $A_2(t) = C \begin{bmatrix} \lambda_1(t) & & 0 \\ & \ddots & \\ 0 & & \lambda_n(t) \end{bmatrix} C^{-1}$ is a path joining CDC^{-1} to I.

$\because$ $A_2(0) = CDC^{-1}$, $A_2(1) = I$

So, the path $\tilde{A} : [0,2] \to GL(n, \mathbb{C})$ def. by

$$\tilde{A}(t) := \begin{cases} A_1(t) : 0 \leq t \leq 1 \\ A_2(t) : 1 \leq t \leq 2 \end{cases}$$

is a path-conn. A to I in $GL(n, \mathbb{C})$.

Ppn: $SL(n, \mathbb{C})$ is path-conn. $\forall\, n \geq 1$.

Pf: If $n=1$, $SL(1, \mathbb{C}) = \{1\}$ which is clearly path-conn.

For $n>1$, the proof is similar to that of $GL(n, \mathbb{C})$.
Let $A_1(t)$ be as before
($\because$ $\det(A_1(t)) = \det A$, we have $A_1(t) \in SL(n, \mathbb{C})$)

For $A_2(t)$, choose $\lambda_1(t), \dots, \lambda_{n-1}(t)$ as before & def.
$\lambda_n(t) = (\lambda_1(t) \dots \lambda_{n-1}(t))^{-1}$, then $\det A_2(t) = 1$ $\forall\, t$

So, path $\tilde{A}(t)$ connects A to I is $SL(n, \mathbb{C})$.

Ppn: $U(n)$ & $SU(n)$ are path-conn. $\forall n \geq 1$.

Pf: Every unitary matrix U can be unitarily diagonalized with eigenvalues of modulus 1.

$$U = U_1 \begin{bmatrix} e^{i\theta_1} & & 0 \\ & \ddots & \\ 0 & & e^{i\theta_n} \end{bmatrix} U_1^{-1}$$

where U_1 unitary, $\theta_i \in \mathbb{R}$.

Def. $$U(t) = U_1 \begin{bmatrix} e^{i\theta_1(1-t)} & & 0 \\ & \ddots & \\ 0 & & e^{i\theta_n(1-t)} \end{bmatrix} U_1^{-1}$$

Then $U(t)$ is a path in $U(n)$ joining U to I.

Hence $U(n)$ is path-conn.

Lie Gp. Homomorphism

$\Phi : G \to H$ called a Lie gp. homomorphism if

1. Φ is a gp. homomp.
2. Φ is cont.

If, in addⁿ, Φ is bijective & the inverse map Φ^{-1} is cont., then Φ is called Lie gp. homomp.
i.e Φ is a Lie gp. homomp. if Φ is both a gp. homomp. as well as a homeomorphism of G & H as topo. sp.

eg: 1. $\det : GL(n, \mathbb{C}) \to \mathbb{C}^*$ is a Lie gp. homomp.

2. $\Phi : \mathbb{R} \to SO(2)$ is a Lie gp. homomp.

$$\theta \mapsto \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Viewing MLGs as real MLGs

In HW 1.5 we defined an inj. map $\rho_n : M_n(\mathbb{C}) \hookrightarrow M_{2n}(\mathbb{R})$ and proved that

1. $\forall A \in M_n(\mathbb{C})$, $\rho_n \circ T_A = T_{\rho_n(A)} \circ \rho_n$
2. $\forall A, B \in M_n(\mathbb{C})$, $\rho_n(AB) = \rho_n(A) \cdot \rho_n(B)$

Note: 1. $A \in GL(n, \mathbb{C})$

$\Leftrightarrow$ $T_A : \mathbb{C}^n \to \mathbb{C}^n$ bijective

$\Leftrightarrow$ $T_{\rho(A)} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ bijective

$\Leftrightarrow$ $\rho(A) \in GL(2n, \mathbb{R})$

$$\rho_n : GL(n, \mathbb{C}) \longleftrightarrow GL(2n, \mathbb{R})$$

So, $GL(n, \mathbb{C})$ is isomorphic to a subgp. of $GL(2n, \mathbb{R})$

2. $q = a + bi + cj + dk \in \mathbb{H}$, we can write

$$q = (a+bi) + (c+di)j$$
$$= z + wj, \text{ where } z, w \in \mathbb{C}$$

So, in a completely analogous way, we can define

$$g_n : \mathbb{H}^n \longrightarrow \mathbb{C}^{2n}$$
$$(z_1 + w_1 j, \dots, z_n + w_n j) \longmapsto (z_1, w_1, \dots, z_n, w_n)$$

&

an inj. map $\Psi_n : M_n(\mathbb{H}) \longrightarrow M_{2n}(\mathbb{C})$ by defining

$$\Psi_1(z+wj) = \begin{bmatrix} z & w \\ -\overline{w} & \overline{z} \end{bmatrix}$$

and constructing Ψ_n out of Ψ_1 similarly to how f_n was constructed from f_1 s.t

1. $\forall A \in M_n(\mathbb{H})$, $g_n \circ T_A = T_{\Psi_n(A)} \circ g_n$
2. $\forall A, B \in M_n(\mathbb{H})$, $\Psi_n(AB) = \Psi_n(A) \cdot \Psi_n(B)$

As above, $A \in GL(n, \mathbb{H})$ iff $\Psi_n(A) \in GL(2n, \mathbb{C})$, so $GL(n, \mathbb{H})$ is isomorphic to a subgp. of $GL(2n, \mathbb{C})$.

Conclusion: All matrix groups are real matrix gps.

$\underline{\text{Ppn}}$: 1. $\rho_n(U(n)) = O(2n) \cap \rho_n(GL(n,\mathbb{C}))$

2. $\psi_n(Sp(n)) = U(2n) \cap \psi_n(GL(n,\mathbb{H}))$

$\underline{\text{Pf}}$: 1. Observe that $\forall A \in M_n(\mathbb{C})$, $\rho_n(A^*) = \rho_n(A)^*$

So, $A \in GL(n,\mathbb{C})$, then $\rho_n(A)\cdot\rho_n(A)^* = \rho_n(A)\cdot\rho_n(A^*) = \rho_n(AA^*)$

So, $A \in U(n)$ iff $\rho_n(A) \in O(2n)$

2. Sim. as 1.

$\underline{\text{Rem}}$: $A = \begin{bmatrix} i & j \\ i & j \end{bmatrix} \in M_2(\mathbb{H})$

$T_A : \mathbb{H}^2 \to \mathbb{H}^2$ is <u>not</u> invertible

yet $\det(T_A) \neq 0$ ($\because$ quaternions do <u>not</u> commute)

Ppn: $SU(2)$ is isomorphic to $Sp(1)$.

Pf: First, note that by prev. discussion,

$$\Psi_1(Sp(1)) = \left\{ \begin{bmatrix} z & w \\ -\bar{w} & \bar{z} \end{bmatrix} : z, w \in \mathbb{C} \text{ s.t } |z|^2 + |w|^2 = 1 \right\}$$
$$:= S$$

is a subgp. of $U(2)$.
(Ψ_1 is inj.)

As discussed in week 1, $S = SU(2)$
So, $\Psi_1 : Sp(1) \to SU(2)$ is a gp. isomp.

The map $\Psi_1 : M_1(\mathbb{H}) \to M_2(\mathbb{C})$ is clearly cont. &

$$\operatorname{img}(\Psi_1) = \left\{ \begin{bmatrix} z & w \\ -\bar{w} & \bar{z} \end{bmatrix} : w, z \in \mathbb{C} \right\}$$
$$:= V$$

V is a subsp. of $M_2(\mathbb{C})$ isomorphic to $\mathbb{C}^2 \simeq \mathbb{R}^4$

So, the map $\Psi_1 : M_1(\mathbb{H}) \to V$ is bij, cont. & the map $\Phi : V \to M_1(\mathbb{H})$ is an inverse for $\Psi_1 : M_1(\mathbb{H}) \to V$

$$\begin{bmatrix} z & w \\ -\bar{w} & \bar{z} \end{bmatrix} \mapsto z + wj$$

& is cont.

So, $\Psi_1 : M_1(\mathbb{H}) \to V$ is a homeomorphism & it restricts to a homeomorphism $\Psi_1 : Sp(1) \to SU(2)$

In summary, $\Psi_1 : Sp(1) \to SU(2)$ is an isomp. of MLG.

Polar decomposition for $SL(n,\mathbb{R})$

Positive definite matrix: A real symm. matrix P is said to be positive or positive definite if $\langle x, Px \rangle > 0 \ \forall \ 0 \neq x \in \mathbb{R}^n$

Given a symm. positive matrix P, $\exists$ orthogonal matrix R s.t $P = RDR^{-1}$ where $D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$, $\lambda_i > 0 \ \forall \ 1 \leq i \leq n$

We can construct a 'square root' for P as follows

$$P^{1/2} = RD^{1/2}R^{-1} = R \begin{bmatrix} \lambda_1^{1/2} & & 0 \\ & \ddots & \\ 0 & & \lambda_n^{1/2} \end{bmatrix} R^{-1}$$

Then $P^{1/2}$ is also symm. & positive.

Ex: $P^{1/2}$ is the unique positive symm. matrix whose square is P.

$\underline{\text{Pp}^n}$: Given $A \in SL(n, \mathbb{R})$, $\exists$ a unique pair (R, P) s.t $R \in SO(n)$ & P is real, symmetric, positive, $\det P = 1$ s.t $A = RP$

$\underline{\text{Pf}}$: First, observe that

- if $A = RP$ as in the statement, then

$$A^T A = P R^T R P = P^2$$

- $A^T A$ is symmetric & positive

$\underline{\text{Existence}}$: So, let us define P as $\quad P = (A^T A)^{1/2}$

Then P is real, symmetric & positive.

Now, define $\quad R = AP^{-1} = A\,[(A^T A)^{1/2}]^{-1}$

Check that R is orthogonal:

$$RR^T = A[(A^T A)^{1/2}]^{-1}[(A^T A)^{1/2}]^{-1} A^T = A(A^T A)^{-1} A^T = I$$

$\underline{\text{Uniqueness}}$: For any pair (R, P) as above, with $A = RP$, $P^2 = A^T A$.

Then, uniqueness of real, symmetric, positive square root of P implies that $R = AP^{-1}$ is also unique.

$\underline{\text{Note}}$: If $P \in M_n(\mathbb{C})$ is self-adjoint, we say P is positive if $\langle x, Px \rangle > 0$
$\forall\, 0 \neq x \in \mathbb{C}^n$

Ppⁿ: Given $A \in SL(n, \mathbb{C})$, $\exists$ a unique pair (U, P) with $U \in SU(n)$ & P is self-adjoint, positive, with $\det P = 1$ s.t $A = UP$

Linear Algebra over $\mathbb{H}$

Warning: Quaternions do not commute

To define a vector sp. V over $\mathbb{H}$, we will let scalars act from the right i.e $v.q$ where $v \in V$ & $q \in \mathbb{H}$.

So, by a linear map $T: V \to W$ where V & W are (right) vector sp. over $\mathbb{H}$, we mean that $\forall v_1, v_2 \in V$ & $q \in \mathbb{H}$, we have that

$$T(v_1 + v_2) = T(v_1) + T(v_2)$$

$$T(v.q) = T(v) \cdot q$$

Let $T: V \to V$ be a linear transformation of $\mathbb{H}$ vector sp. Suppose $\{v_i\}_{i=1}^{n}$ is a basis for V & $A = (a_{ij}) \in M_n(\mathbb{H})$ is the matrix whose entries are defined by

$$T(v_i) = \sum_{j=1}^{n} v_j a_{ji}$$

Then if $v = \sum_{i=1}^{n} v_i b_i \in V$ & $b = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$, where $b_i \in \mathbb{H}$ $\forall\ i = 1, \dots, n$.

Then by linearity, we have

$$T(v) = \sum_{i=1}^{n} T(v_i) b_i = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} v_j a_{ji} \right) b_i$$

$$= \sum_{j=1}^{n} v_j \left(\sum_{i=1}^{n} a_{ji} b_i \right)$$

That is, $T(v)$ equals the vector Ab in basis $\{v_i\}_{i=1}^{n}$

As a result, composition of two $\mathbb{H}$-linear maps corresponds to the usual matrix multip^{n}.

<u>Warning</u>: The usual def^{n} of determinant does <u>not</u> behave well, in general

$$\det(AB) \neq \det(A) \cdot \det(B).$$

We can define $GL(n, \mathbb{H})$ as

$$GL(n, \mathbb{H}) = \{ A \in M_n(\mathbb{H}) : \exists B \text{ s.t } AB = BA = I \}$$

Correction: This is with regards to our earlier def^n of ψ_n

Write $q \in \mathbb{H}$ as $q = z + wj$ where $z, w \in \mathbb{C}$ & set

$$\psi_1(z + wj) = \begin{bmatrix} z & -\overline{w} \\ w & \overline{z} \end{bmatrix}$$

and define ψ_n out of ψ_1 in the same way as before.

Also, correct $g_n : \mathbb{H}^n \to \mathbb{C}^{2n}$ to read:

$$(z_1 + w_1 j, \ldots, z_n + w_n j) \mapsto (z_1, w_1, \ldots, z_n, w_n)$$

This gives an embedding $GL(n, \mathbb{H}) \hookrightarrow GL(2n, \mathbb{C})$.

Lie Algebra associated to a Lie gp. G

Let $S \subseteq \mathbb{R}^k$.

We say $\gamma : (-\epsilon, \epsilon) \to S$ is a diff. path in S

if $\gamma(t) = (\gamma_1(t), \dots, \gamma_k(t))$ in terms of coordinates, then

each $\gamma_i : (-\epsilon, \epsilon) \to \mathbb{R}$ is a diff. fn^n.

Let $S \subseteq \mathbb{R}^k$ be a subset & let $p \in S$.

The tangent space to S at p is:

$$T_pS = \{ \gamma'(0) : \gamma : (-\epsilon, \epsilon) \to S \text{ is a diff. path with } \gamma(0) = p \}$$

The Lie algebra to a matrix Lie gp. $G \subseteq GL(n, \mathbb{C})$ is the tangent sp. to G at I. It is denoted by

$$\mathfrak{g} = \mathfrak{g}(G) = T_I G$$

Not^n: To denote Lie algebras, we usually use 'fraktur' letters

We will prove that the Lie alg. of a matrix Lie gp. G is a real vector sp.

Pp^n: (Product rule)

If $\alpha, \beta : (-\epsilon, \epsilon) \to M_n(\mathbb{K})$ where $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ are diff. paths, then the product path

$$(\alpha \cdot \beta)(t) = \alpha(t) \bullet \beta(t)$$

product in $M_n(\mathbb{K})$

is diff. as well & we have

$$(\alpha \cdot \beta)'(t) = \alpha'(t) \cdot \beta(t) + \alpha(t) \cdot \beta'(t)$$

Pf: If $n=1$ & $\mathbb{K} = \mathbb{R}$, this is just the product rule from single-variable calculus.

If $n=1$, $\mathbb{K} = \mathbb{C}$, say $\alpha(t) = a(t) + b(t)i$
& $\beta(t) = c(t) + d(t)i$

Then, $(\alpha \cdot \beta)(t) = [(a+bi)\cdot(c+di)]'$

$$= (ac-bd)' + (bc+ad)'i$$

$$= (a'c + ac' - b'd - bd') + (b'c + bc' + a'd + ad')i$$

$$= (a'c - b'd) + (b'c + a'd)i + (ac' - bd') + (bc' + ad')i$$

$$= \alpha' \cdot \beta + \alpha \cdot \beta'$$

Sim., for $n=1$, $\mathbb{K} = \mathbb{H}$.

In general,

$$((\alpha \cdot \beta)(t))_{ij} = \sum_{l=1}^{n} \alpha_{il}(t) \beta_{lj}(t)$$

and,

$$((\alpha \cdot \beta)'(t))_{ij} = \sum_{l=1}^{n} \alpha'_{il}(t) \beta_{lj}(t) + \alpha_{il}(t) \beta'_{lj}(t)$$

$$= (\alpha'(t) \cdot \beta(t))_{ij} + (\alpha(t) \cdot \beta'(t))_{ij}$$

If $\gamma : (-\epsilon, \epsilon) \to GL(n, \mathbb{C})$ is a diff. path, so is the inverse path $t \mapsto \gamma^{-1}(t)$.

T'fore by product rule,

$$0 = \frac{d}{dt}(I) = \frac{d}{dt}(\gamma(t) \cdot \gamma^{-1}(t)) = \frac{d}{dt}\gamma(t) \cdot \gamma^{-1}(t) + \gamma(t) \frac{d}{dt}\gamma^{-1}(t)$$

So, if $\gamma(0) = I$, we get

$$\left.\frac{d}{dt}(\gamma^{-1}(t))\right|_{t=0} = -\left.\frac{d}{dt}(\gamma(t))\right|_{t=0}$$

i.e the inverse of a path goes through I in the opposite dirn.

<u>Ppn</u>: The Lie alg. of a MLG $G \subseteq GL(n,\mathbb{C})$ is a real subsp. of $M_n(\mathbb{C})$

<u>Pf</u>: Let $\lambda \in \mathbb{R}$ & $A \in \mathfrak{g}$ i.e $A = \gamma'(0)$ for some diff. path $\gamma(t)$ in G with $\gamma(0) = I$.

The path $\sigma(t) = \gamma(\lambda t)$ has initial velocity vector

$$\sigma'(0) = \lambda\gamma'(0) = \lambda A.$$

So, $\lambda A \in \mathfrak{g}$

Now, let $A, B \in \mathfrak{g}$ i.e $A = \gamma'(0)$ & $B = \beta'(0)$ for some diff. paths, γ, β in G, with $\gamma(0) = I$ & $\beta(0) = I$.

The product path $\sigma(t) = \gamma(t) \cdot \beta(t)$ is diff. & lies in G
& satisfies $\sigma(0) = I$.

Product rule $\Rightarrow$ $\sigma'(0) = A + B$
So, $A + B \in \mathfrak{g}$

This proves that $\mathfrak{g}$ is a real subsp. of $GL(n, \mathbb{C})$.

<u>Dimension</u>: The dimension of MLG G is defined as the dimension of its Lie alg. (as a real vector sp.).

Examples of Lie Alg.

Not^n: We will use lowercase letters to denote Lie algebras

eg: $\mathfrak{gl}(n,\mathbb{R})$ is the Lie alg. of $GL(n,\mathbb{R})$.

Pp^n: $\mathfrak{gl}(n,\mathbb{K}) = M_n(\mathbb{K})$. In particular $\dim GL(n,\mathbb{R}) = n^2$
& $\dim GL(n,\mathbb{C}) = 2n^2$

Pf: Certainly $\mathfrak{gl}(n,\mathbb{K}) \subseteq M_n(\mathbb{K})$

Let $A \in M_n(\mathbb{K})$, then $\gamma_A : \mathbb{R} \to M_n(\mathbb{K})$ is a diff. path in $M_n(\mathbb{K})$
$$t \mapsto I + tA$$

& by continuity of det, $\exists\, \epsilon > 0$ s.t $\gamma_A : (-\epsilon, \epsilon) \to M_n(\mathbb{K})$
is a diff. path in $GL_n(\mathbb{K})$

Also, $\gamma_A(0) = I$ & $\gamma_A'(0) = A$. This shows that $A \in T_I GL_n(\mathbb{K})$
ie $A \in \mathfrak{gl}(n,\mathbb{K})$

$$\Rightarrow M_n(\mathbb{K}) \subseteq \mathfrak{gl}(n,\mathbb{K})$$

$\underline{\text{Ppn}}$: The Lie Alg. $u(1)$ of $U(1)$ equals $\text{span}\{[i]\}$. So $\dim U(1) = 1$

$\underline{\text{Pf}}$: The path $\gamma(t) = [e^{it}]$ lies in $U(1)$ & $\gamma(0) = I$.
Since $\gamma'(0) = i$, so $\text{span}\{[i]\} \subseteq U(1)$.

Now, let $\alpha(t) = [a(t) + ib(t)]$ be a diff. path in $U(1)$
with $\alpha(0) = 1$.
Since $|\alpha(t)|^2 = a(t)^2 + b(t)^2 = 1$

So, in particular, $a(t) \leq 1$. Hence $a(0) = 1$ must be local max
of $a(t)$, so $a'(0) = 0$
So, $\alpha'(0) = b'(0) i \in \text{span}\{[i]\}$

$so(n) = \{A \in M_n(\mathbb{K}) : A + A^T = 0\}$ is called the set of skew-symm. matrices in $M_n(\mathbb{R})$.

- $so(n)$ is real subspace of $M_n(\mathbb{R})$
- $O(n) = SO(n) \sqcup O^-(n)$

Thm: The lie alg. of $O(n)$ & $SO(n)$ is $\mathfrak{so}(n)$.

Pf: $\gamma : (-\epsilon, \epsilon) \to O(n)$ is a diff. path with $\gamma(0) = I$
& $\gamma(t) \cdot \gamma(t)^T = I$

$$\Rightarrow \gamma'(t)\gamma(t)^T + \gamma(t) \cdot \gamma'(t)^T = 0$$

$$\Rightarrow \gamma'(0) + \gamma'(0)^T = 0$$

T'fore lie alg. of $O(n) \subseteq \mathfrak{so}(n)$

For reverse inclusion, it is sufficient to construct a path in $O(n)$ in dirxn of each elem. of $\mathfrak{so}(n)$.

$\because$ $\mathfrak{so}(n)$ is a vector sp., it is sufficient to construct a basis.

Observe $\mathcal{B} = \{ E_{ij} - E_{ji} : 1 \leq i < j \leq n \}$ forms a basis of $\mathfrak{so}(n)$.

The path $\gamma_{ij}(t) = I + \sin(t)(E_{ij} - E_{ji}) + (-1 + \cos(t))(E_{ii} + E_{jj}) \in SO(n)$ $\forall t$

$$\gamma_{ij}(0) = I$$

$$\gamma'_{ij}(0) = E_{ij} - E_{ji}$$

Cor: $\dim(O(n)) = n(n-1)/2$

Lem: Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. If $\gamma : (-\epsilon, \epsilon) \to M_n(\mathbb{K})$ is diff.
& $\gamma(0) = I$, then

$$\left.\frac{d}{dt}\right|_{t=0} \det(\gamma(t)) = \operatorname{trace}(\gamma'(0))$$

Pf: (Not^n)

If A is an $n \times n$ matrix, we will write $A[i\ j]$ to mean the $(n-1)\times(n-1)$ matrix obtained by deleting the ith row & jth col of A.

Then $$\det \gamma(t) = \sum_{j=1}^{n} (-1)^{j+1} \gamma(t)_{1j} \det(\gamma(t)[1\ j]) = \gamma'(0).$$

Differentiating w.r.t t,

$$\left.\frac{d}{dt}\right|_{t=0} \det(\gamma(t)) = \sum_{j=1}^{n} (-1)^{1+j} \gamma'(0)_{ij} \det(\gamma(t)[1\ j])$$

$$+ \gamma(0)_{1j} \left.\frac{d}{dt}\right|_{t=0} \det(\gamma(t)[1\ j])$$

$$= \gamma'(0)_{11} + \left.\frac{d}{dt}\right|_{t=0} \det(\gamma(t)[1\ 1])$$

$$\vdots$$

$$= \gamma'(0)_{11} + \dots + \gamma'(0)_{nn} = \operatorname{trace}(\gamma'(0))$$

Thm: Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. The Lie Algebra $\mathfrak{sl}(n, \mathbb{K})$ of $SL(n, \mathbb{K})$ is

$$\mathfrak{sl}(n, \mathbb{K}) = \{A \in M_n(\mathbb{K}) : \text{trace}(A) = 0\}$$

Pf: If $\gamma : (-\epsilon, \epsilon) \to SL(n, \mathbb{K})$ is a diff. path with $\gamma(0) = I$. Then $\det(\gamma(t)) = 1$, so using the lemma we just proved, we see that $\text{trace}(\gamma'(0)) = 0$.

So, $\mathfrak{sl}(n, \mathbb{K}) \subseteq \{A \in M_n(\mathbb{K}) : \text{trace}(A) = 0\}$

Now, let $A \in M_n(\mathbb{K})$ s.t $\text{trace}(A) = 0$.
For ϵ small enough, the path $\alpha : (-\epsilon, \epsilon) \to M_n(\mathbb{K})$ def. as $\alpha(t) = I + tA$ lies in $GL(n, \mathbb{K})$ & $\alpha'(0) = A$.
But $\alpha(t)$ need <u>not</u> have det 1.

So, def. $\gamma(t)$ to be the path obtained by multiplying the first row of $\alpha(t)$ by $1/\det(\alpha(t))$.
& leaving other rows unchanged.

Then $\gamma'(t)$ is a diff. path in $SL(n, \mathbb{K})$ with $\gamma(0) = I$.

Also, $\gamma'(t)_{ij} = \alpha'(t)_{ij}$ for $2 \leq i \leq n$, $1 \leq j \leq n$ &

$$\gamma'(0)_{ij} = \alpha'(0)_{ij} - \underbrace{\text{trace}(\alpha'(0))}_{0} \cdot \alpha(0)_{ij} = \alpha'(0)_{ij}$$

All in all, this yields that $\gamma'(0) = \alpha'(0) = A$

$\Rightarrow \{A \in M_n(\mathbb{K}) : \text{trace}(A) = 0\} \subseteq sl(n, \mathbb{K})$

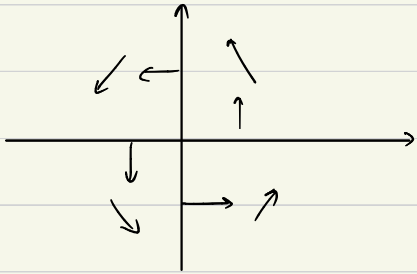
Lie Algebra as a Vector field

A vector field is a continuous fxn $F : \mathbb{R}^m \to \mathbb{R}^m$.

If we picture $F(v)$ as a vector drawn at the pt $v \in \mathbb{R}^m$, we can think of F as associating a vec. to each pt. of $\mathbb{R}^m$.

If $A \in M_n(\mathbb{K})$, then $T_A : \mathbb{K}^n \to \mathbb{K}^n$ is a vector field on $\mathbb{K}^n$

eg: The vector field associated to $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is



A diff. path $\gamma : (-\epsilon, \epsilon) \to GL(n, \mathbb{K})$ gives rise to a <u>one-parameter family of linear transformations</u> on $\mathbb{K}^n$ given by $T_{\gamma(t)}$

If we fix $X \in \mathbb{K}^n$, consider

$$\sigma(t) = T_{\gamma(t)}(X)$$

σ is a diff. path in $\mathbb{K}^n$.

If $\gamma(0) = I$, then $\sigma(0) = X$. By the product rule (true also for non-square matrix)

$$\sigma'(0) = T_{\gamma'(0)}(X)$$

We can think of $T_{\gamma'(0)}$ as a vector field on $\mathbb{K}^n$ whose value at any $X \in \mathbb{K}^n$ gives the direction in which X is initially measured by the family of linear transformations $T_{\gamma(t)}$.

eg: Consider the path $\gamma(t) = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$ in $SO(2)$.

Its initial tangent vector is $A = \gamma'(0) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ lies in the Lie alg. $so(2)$ of $SO(2)$.

Infact $so(2) = \text{span}\{A\}$

Q. What is the 'best'/'most natural' diff. path $\alpha(t)$ in $SO(2)$ with $\alpha(0) = I$ & has $\alpha'(0) = A$.

A. Every path in $SO(2)$ through I, with tangent vec. A, looks like $\alpha(t) = \begin{bmatrix} \cos f(t) & -\sin f(t) \\ \sin f(t) & \cos f(t) \end{bmatrix}$

What is special about the choice $f(t) = t$?

For every $X \in \mathbb{R}^2$, the path $\beta(t) = T_{\gamma(t)} X$ in $\mathbb{R}^2$, is an integral curve of the vec. field T_A.

<u>Integral curve</u>: A path $\beta : (-\epsilon, \epsilon) \to \mathbb{R}^n$ is called an integral curve of a vec. field $F : \mathbb{R}^m \to \mathbb{R}^m$ if $\beta'(t) = F(\beta(t))$ $\forall\, t \in (-\epsilon, \epsilon)$

For $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, the integral curve of the vec. field T_A are circles centered at the origin, parametrized to travel counterclockwise (with speed one radian per unit time)

More generally, if $A \in \mathfrak{gl}(n, k)$, we want to find the 'best' path inside $GL(n, \mathbb{K})$ with $\gamma(0) = I$ & $\gamma'(0) = A$.

It turns out that such a path can be obtained as "$t \mapsto e^{tA}$" where the "matrix exponential"

$$e^X = \sum_{k=0}^{\infty} \frac{X^k}{k!}$$

Matrix exponentiation

Last week, while describing the Lie alg. of some familiar mlg G, we constructed some explicit path in each G, in the direction of some given $A \in M_n(\mathbb{K})$.

eg: - $gl(n, \mathbb{K})$, $A \in M_n(\mathbb{K})$

$$\gamma(t) = I + tA$$

- $so(3)$, $A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} = E_{13} - E_{31}$

$$\gamma(t) = \begin{pmatrix} \cos(t) & 0 & \sin(t) \\ 0 & 1 & 0 \\ -\sin(t) & 0 & \cos(t) \end{pmatrix}$$

$T_{\gamma(t)}$ rotates the subsp. span$\{e_1, e_3\}$ by angle $-t$ & leaves e_2 fixed.

- $sl(n, \mathbb{K})$: started with $\alpha(t) = I + tA$ & then rescaled the first row of $\alpha(t)$ appropriately.

Given $A \in gl(n, \mathbb{K})$, is there a 'best' path $\gamma(t) \subseteq GL(n, \mathbb{K})$ s.t $\gamma(0) = I$ & $\gamma'(0) = A$?

One potential notion of 'best' path could be $\gamma(t)$ s.t for $X \in \mathbb{K}^n$, the path $\beta(t) = T_{\gamma(t)}(X)$ (β is a path in $\mathbb{K}^n$) is an integral curve for the vector field on $\mathbb{K}^n$ defined by $p \in \mathbb{K}^n \to A \cdot p$

Another possible interpretation of a 'good' path $\gamma(t)$ in $GL(n, \mathbb{K})$ with $\gamma(0) = I$ & $\gamma'(0) = A$:
If $G \subseteq GL(n, \mathbb{K})$ is mlg & $A \in gl(n, \mathbb{K}) = M_n(\mathbb{K})$ & if A lies in $g = T_I G$, then, is this the 'best' path guaranteed to be contained in G ?

To answer these questions, we will construct paths defined in terms of matrix exponentiation; we will want to make sense of the power series

$$e^X = \sum_{k=0}^{\infty} \frac{X^k}{k!} \quad \text{where } X \in M_n(\mathbb{K})$$

- Series in $\mathbb{K}$:

We will consider the series of the form $\sum_{k=0}^{\infty} a_k$ with each $a_k \in \mathbb{K}$.

Ppn:

1. If $\sum a_k$ conv. absolutely, it conv.

2. If $\sum a_k = A$ & $\sum b_n = B$, then $\sum (a_n + b_n) = A + B$ & $\sum c a_n = cA$ for a fixed c.

3. Suppose $\sum a_n$ conv. abs., $\sum a_n = A$ & $\sum b_n = B$ & def. $c_n = \sum_{k=0}^{n} a_k b_{n-k}$, then $\sum c_n = AB$

4. For any power-series $\sum c_n x^n$, $\exists\, R \in [0, \infty]$, called its radius of conv. s.t the series conv. for $|x| < R$ & diverges if $|x| > R$

   R can be computed as $R := \left(\limsup_{n \to \infty} |c_n|^{1/n} \right)^{-1}$

5. Let $f(x) = \sum_{n=0}^{\infty} c_n x^n$ be a power series with radius of conv. R. The restriction $f : (-R, R) \to \mathbb{K}$ is a diff. path in $\mathbb{K}$ with derivative $f'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1}$

- Series in $M_n(\mathbb{K})$:

We say that a series $\sum A_\ell = A_0 + A_1 + \dots$ of elems $A_\ell \in M_n(\mathbb{K})$ conv. (abs.) if for each i,j, the series $(A_0)_{ij} + (A_1)_{ij} + \dots$ conv. (abs.) to some $A_{ij} \in \mathbb{K}$.

In this case, we write $\sum A_\ell = A$

Ppn: Suppose $\sum A_\ell$ conv. abs. Suppose $\sum A_\ell = A$, $\sum B_\ell = B$.
Let $C_\ell := \sum_{k=0}^{\ell} A_k B_{\ell-k}$. Then $\sum C_\ell = A \cdot B$

For $A \in M_n(\mathbb{K})$, we will write $\|A\|$ for the Euclidean norm on $M_n(\mathbb{K})$ regarded as $\mathbb{R}^{n^2}$ (or $\mathbb{R}^{2n^2}$)

Ppn: For all $X, Y \in \mathbb{K}^n$, $|\langle X, Y\rangle| \leq \|X\| \cdot \|Y\|$

Pf: Let $X, Y \in \mathbb{K}^n$, let $\alpha = \langle X, Y\rangle$.
Assume $X \neq 0$.
For $\lambda \in \mathbb{K}$, we have

$$
\begin{aligned}
0 &\leq \|\lambda X + Y\|^2 \\
&= \langle \lambda X + Y, \lambda X + Y\rangle \\
&= \lambda^2 \langle X, X\rangle + \lambda \langle X, Y\rangle \\
&\quad + \bar{\lambda}\langle Y, X\rangle + \langle Y, Y\rangle \\
&= |\lambda|^2 \|X\|^2 + 2\operatorname{Re}(\lambda\langle X, Y\rangle) + \|Y\|^2
\end{aligned}
$$

Choosing $\lambda = \dfrac{-\bar{\alpha}}{\|X\|^2}$ gives,

$$0 \le \frac{|\alpha|^2}{\|X\|^2} - \frac{2|\alpha|^2}{\|X\|^2} + \|Y\|^2 \Rightarrow |\alpha| \le \|X\| \cdot \|Y\|$$

<u>Lem</u>: for all $X, Y \in M_n(\mathbb{K})$, $\|XY\| \le \|X\| \cdot \|Y\|$

<u>Pf</u>: for each pair of indices i, j,

$$|(XY)_{ij}|^2 = \left| \sum_{l=1}^{n} X_{il} Y_{lj} \right|^2$$
$$= |\langle (\text{Row } i \text{ of } X), (\text{Col. } j \text{ of } \bar{Y})^T \rangle|^2$$

$$\le \|\text{Row } i \text{ of } X\| \cdot \|(\text{Col. } j \text{ of } \bar{Y})^T\|$$
$$= \left(\sum_{l=1}^{n} |X_{il}|^2\right) \cdot \left(\sum_{l=1}^{n} |Y_{lj}|^2\right)$$

Summing over all i, j

$$\|XY\|^2 = \sum_{i,j=1}^{n} |(XY)_{ij}|^2 \le \sum_{i,j=1}^{n} \left(\sum_{l=1}^{n} |X_{il}|^2\right) \cdot \left(\sum_{l=1}^{n} |Y_{lj}|^2\right)$$

$$= \left(\sum_{i,j=1}^{n} |X_{ij}|^2\right)\left(\sum_{i,j=1}^{n} |Y_{ij}|^2\right) = \|X\|^2 \cdot \|Y\|^2$$

$$\Rightarrow \|XY\| \le \|X\| \cdot \|Y\|$$

If $f(x) = \sum_{n=0}^{\infty} c_n x^n$ is a power series with coeffs. $c_i \in \mathbb{K}$, we can evaluate f on a matrix $A \in M_n(\mathbb{K})$.

The result is a series in $M_n(\mathbb{K})$.

$$f(A) = c_0 I + c_1 A + c_2 A^2 + \dots$$

<u>Ppn</u>: Let $\sum_{n=0}^{\infty} c_n x^n$ be a power series with coeffs $c_i \in \mathbb{K}$, with radius of conv. R. If $A \in M_n(\mathbb{K})$ with $\|A\| < R$ then,

$f(A) = c_0 I + c_1 A + c_2 A^2 + \dots$ conv. abs.

<u>Pf</u>: For any pair of indices i, j, we need to prove that

$$|(c_0 I)_{ij}| + |(c_1 A)_{ij}| + |(c_2 A^2)_{ij}| + \dots \text{ conv.}$$

The l^{th} term of this series satisfies

$$\begin{aligned} |(c_l A^l)_{ij}| &\leq \|c_l A^l\| = |c_l| \, \|A^l\| \\ &\leq |c_l| \, \|A\|^l \end{aligned}$$

Since $\|A\| < R$ = radius of conv. of f, the result follows.

Matrix exponentiation:

$$e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \dots$$

Since the ROC of the power series

$$f(x) = e^x = \sum_{l=0}^{\infty} \frac{x^l}{l!} \quad \text{is } \infty,$$

Therefore by proposition, e^A conv. abs. $\forall A \in M_n(\mathbb{K})$

Ppn: Let $A \in gl(n, \mathbb{K})$. The path $\gamma : \mathbb{R} \to M_n(\mathbb{K})$ def. by $\gamma(t) = e^{tA}$ is diff. & $\gamma'(t) = A\gamma(t) = \gamma(t)A$

Pf: Consider $\gamma(t) = e^{tA} = I + tA + \frac{t^2A^2}{2} + \dots$

Each entry of the matrix $\gamma(t)$ is a power series in the variable t, which can be diff. term-by-term, to give

$$\gamma'(t) = 0 + A + tA^2 + \frac{1}{2}t^2A^3 + \dots = A\gamma(t) = \gamma(t)A$$

depending on whether you factor out A from left or right.

Cor: Let $A \in M_n(\mathbb{K})$ & let $\gamma(t) = e^{tA}$. Then, for each $X \in \mathbb{R}^n$, the path $\alpha(t) = T_{\gamma(t)}(X)$ is an integral curve of the vector field T_A on $\mathbb{K}^n$.

Pf: $\alpha(t) = \gamma(t)X$ so
$$\begin{aligned}\alpha'(t) &= \gamma'(t)X \\ &= A\gamma(t)X \\ &= A\alpha(t) \\ &= T_A(\alpha(t))\end{aligned}$$

Cor: Let $A \in M_n(\mathbb{K})$ & let $\gamma(t) = e^{tA}$. Then $\gamma(t)$ is an integral curve of the following vector field V on $M_n(\mathbb{K})$ whose value at $g \in M_n(\mathbb{K})$ is $V(g) = A \cdot g$

Pf: $\gamma'(t) = A \cdot \gamma(t) = V(\gamma(t))$

Rem: We are yet to prove that $\gamma(t) = e^{tA}$ is a path in $GL(n, \mathbb{K})$.

Ppn: If $AB = BA$, then $e^{A+B} = e^A \cdot e^B$

Pf: $$e^A \cdot e^B = \left(\sum_{l=0}^{\infty} \frac{A^l}{l!}\right)\left(\sum_{l=0}^{\infty} \frac{B^l}{l!}\right)$$
$$= \sum_{l=0}^{\infty} \left(\sum_{k=0}^{l} \frac{A^k B^{l-k}}{k!(l-k)!} \right)$$
$$= \sum_{l=0}^{\infty} \frac{1}{l!} \underbrace{\left(\sum_{k=0}^{l} \binom{l}{k} A^k B^{l-k} \right)}_{(A+B)^l} \quad (\because AB = BA)$$
$$= \sum_{l=0}^{\infty} \frac{(A+B)^l}{l!}$$
$$= e^{A+B}$$

Ppn: For any $A \in M_n(\mathbb{K})$, $e^A \in GL(n, \mathbb{K})$.

T'fore, matrix exponentiation is a map
$$\exp : \mathfrak{gl}(n, \mathbb{K}) \to GL(n, \mathbb{K})$$

Pf: Since A & $-A$ commute, we have

$$e^A \cdot e^{-A} = e^{A-A} = e^0 = I$$

So, e^A is an invertible matrix, with inverse e^{-A}.

Ppⁿ: For $A \in M_n(\mathbb{K})$, $(e^A)^* = e^{A^*}$

Pf: $e^A = I + A + \frac{A^2}{2} + \frac{A^3}{6} + \ldots$

The map $f : M_n(\mathbb{K}) \to M_n(\mathbb{K})$ is cts.

$$A \mapsto A^*$$

Hence, $$f\left(\lim_{n\to\infty} \sum_{l=0}^{n} \frac{A^l}{l!}\right) = \lim_{n\to\infty} f\left(\sum_{l=0}^{n} \frac{A^l}{l!}\right)$$

$$= \lim_{n\to\infty} \sum_{l=0}^{n} \frac{(A^*)^l}{l!} \quad (\because (A^l)^* = (A^*)^l)$$

$$= e^{A^*}$$

i.e $(e^A)^* = e^{A^*}$

Ppⁿ: If $A \in so(n)$, then $e^A \in O(n)$.

Pf: Since $A \in so(n)$, $A^T = -A$. T'fore

$$e^A \cdot (e^A)^T = e^A \cdot e^{A^T} = e^A \cdot e^{-A} = e^{A-A} = e^0 = I$$

Ppⁿ: If $A \in u(n)$, then $e^A \in U(n)$.

Pf: Since $A \in u(n)$, $A^* = -A$. T'fore

$$e^A \cdot (e^A)^* = e^A \cdot e^{A^*} = e^A \cdot e^{-A} = e^{A-A} = e^0 = I$$

Lem: Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. For any $A \in M_n(\mathbb{K})$.

$$\det(e^A) = e^{\text{trace}(A)}$$

Pf: Let $f(t) = \det(e^{tA})$

$$f'(t) = \lim_{h \to 0} \frac{1}{h} \left(\overbrace{\det(e^{(t+h)A})}^{\det(e^{tA}) \cdot \det(e^{hA})} - \det(e^{tA}) \right)$$

$$= \lim_{h \to 0} \frac{1}{h} \left(\det(e^{tA}) \left(\det(e^{hA}) - 1 \right) \right)$$

$$= \det(e^{tA}) \lim_{h \to 0} \frac{\det(e^{hA}) - 1}{h}$$

$$= \det(e^{tA}) \left. \frac{d}{dt} \right|_{t=0} \left(\det(e^{tA}) \right) = \det(e^{tA}) \, \text{trace}(A)$$

So, if $f'(t) = f(t) \, \text{trace}(A)$ & $f(0) = 1$

So, the unique sol^n of this diff. eq^n in f is:

$$f(t) = e^{t \cdot \text{trace} A}$$

So, in particular, $f(1) = e^{\text{trace} A}$ which completes the proof.

$\underline{Pp^n}$: The Lie alg. $su(n)$ of the mlg $SU(n)$ is given by

$$su(n) = u(n) \cap sl(n, \mathbb{C})$$
$$= \{ A \in M_n(\mathbb{C}) : A + A^* = 0 \ \& \ \text{trace}(A) = 0 \}$$

$\underline{Pf}$: Since $SU(n) = U(n) \cap SL(n, \mathbb{C})$

$A \in su(n) = T_I SU(n)$ implies

$A \in u(n)$ & $A \in sl(n, \mathbb{C})$ i.e we have

$su(n) \subseteq u(n) \cap sl(n, \mathbb{C})$

For the other inclusion, let $A \in M_n(\mathbb{C})$ s.t $A + A^* = 0$ & $\text{trace}(A) = 0$.

Then the path $\gamma(t) = e^{tA}$ satisfies

1. $\gamma(t) \subseteq U(n)$ (by prev. pp^n)
2. $\gamma(t) \subseteq SL(n, \mathbb{C})$ (lemma implies that $\det(e^{tA}) = e^{t \cdot \text{trace}(A)} = e^0 = 1$)

so, $\gamma(t)$ is a path in $SU(n)$ with $\gamma(0) = I$ & $\gamma'(0) = A$.

This shows that $A \in su(n)$.

Rem: 1. If $A \in \mathfrak{so}(n)$, then $\gamma(t) = e^{tA}$ is a path in $O(n)$.

2. If $A \in \mathfrak{u}(n)$, then $\gamma(t) = e^{tA}$ is a path in $U(n)$.

3. If $A \in \mathfrak{su}(n)$, then $\gamma(t) = e^{tA}$ is a path in $SU(n)$.

4. If $A \in \mathfrak{gl}(n, \mathbb{K})$, then $\gamma(t) = e^{tA}$ is a path in $GL(n, \mathbb{K})$.

Def: A one-parameter group in a mlg G, is a differentiable group homomorphism

$$\gamma : (\mathbb{R}, +) \to G$$

Ppn : 1. For any $A \in gl(n, \mathbb{K})$, $\gamma(t) = e^{tA}$ is a one-parameter gp.

2. Every one-parameter gp. in $GL(n, \mathbb{K})$ has the description $\gamma(t) = e^{tA}$ for some $A \in gl(n, \mathbb{K})$

Pf : 1. $\gamma(t_1 + t_2) = e^{t_1 A + t_2 A} = e^{t_1 A} \cdot e^{t_2 A} = \gamma(t_1) \cdot \gamma(t_2)$

In particular, $\gamma(t) \cdot \gamma(-t) = I$

$$\gamma^{-1}(t) = \gamma(-t)$$

2. Suppose $\gamma(t)$ is a one-parameter gp. in $GL(n, \mathbb{K})$.
Let $A = \gamma'(0)$.

Then, for $t \in \mathbb{R}$,

$$\gamma'(t) = \lim_{h \to 0} \frac{1}{h} \left(\gamma(t+h) - \gamma(t) \right)$$

$$= \lim_{h \to 0} \frac{1}{h} \left(\gamma(t) \cdot \gamma(h) - \gamma(t) \right)$$

$$= \gamma(t) \left(\lim_{h \to 0} \frac{\gamma(h) - I}{h} \right) = \gamma'(0)\, \gamma(t) = A \gamma(t)$$

Then $\frac{d}{dt}(e^{-tA}\gamma(t)) = e^{-tA}\gamma'(t) + \frac{d}{dt}(e^{-tA})\gamma(t)$

$$= e^{-tA}A\gamma(t) - e^{-tA}A\gamma(t) = 0.$$

$\Rightarrow e^{-tA}\gamma(t)$ is constant in t, so

$\forall\, t \in \mathbb{R}$, $e^{-tA}\cdot\gamma(t) = e^{-0}\gamma(0) = I \Rightarrow \gamma(t) = e^{tA}$

Ppn: For all $A, B \subseteq M_n(\mathbb{K})$ with A inverible,

$$e^{ABA^{-1}} = Ae^{B}A^{-1}$$

Pf: $Ae^{B}A^{-1} = A\left(I + B + \frac{B^2}{2} + \frac{B^3}{6} \cdots\right)A^{-1}$

$$= I + ABA^{-1} + \frac{1}{2}\underbrace{AB^2A^{-1}}_{(ABA^{-1})^2} + \frac{1}{6}\underbrace{AB^3A^{-1}}_{(ABA^{-1})^3} + \ldots$$

Review of Multivariable Calculus

Let $U \subseteq \mathbb{R}^n$ be an open set.
Any fn $f: U \to \mathbb{R}^m$ can be written as $f = (f_1, \ldots, f_m)$
where each $f_i : U \to \mathbb{R}$.

Defn: Let $p \in U$ & let $v \in \mathbb{R}^n$. The directional derivative of f in the direction v, at the pt. p is defined as

$$f'(p;v) = \lim_{t \to 0} \frac{f(p+tv) - f(p)}{t}$$

if this limit exists.

Geometric interpretation:
Let $\gamma(t) = p + tv$ be a straight path in $\mathbb{R}^n$.
Then $f \circ \gamma$ is a path in $\mathbb{R}^m$.
If the initial velocity vector of the path $f \circ \gamma$ in $\mathbb{R}^m$ exists, it is called $f'(p;v)$.

Def^n: The directional derivatives of the component fx^{n}s $\{f_1, \dots, f_m\}$ in the directions of the standard basis vecs. $\{e_1, \dots, e_n\}$ in $\mathbb{R}^n$, are called the partial derivatives of f & are denoted:

$$D_j f_i(p) = \frac{\partial f_i(p)}{\partial x_j} = f_i'(p, e_j)$$

$$= \lim_{t \to 0} \frac{f_i(p + te_j) - f(p)}{t}$$

Def^n: Fix i, j. If $\frac{\partial f_i}{\partial x_j}(p)$ exists $\forall \, p \in U$, then

$$p \mapsto \frac{\partial f_i(p)}{\partial x_j}$$ is another fx^n from U to $\mathbb{R}$.

Its partial derivatives (if they exist) are called second order partial derivatives of f.

Sim., we can define third order partial derivatives, or more generally, partial derivatives of order r, where $r \in \mathbb{Z}_{>0}$

Defⁿ: The fxⁿ f is said to be of class C^r on U if all the partial derivatives of each f_i, of order $\leq r$ exist & are cts. on U.

The fxⁿ f is said to be of class C^∞ on U, if f is of class C^r, for each $r \in \mathbb{Z}_{>0}$.

Defⁿ: $f: U \to \mathbb{R}^m$ is said to be differentiable at $p \in U$ if $\exists$ a linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ s.t

$$\lim_{h \to 0} \frac{\| f(p+h) - f(p) - T(h) \|}{\|h\|} = 0$$

If f is diff. at p, then $Df_p := T$ is called the derivative of f at p.

Thm: If $f: U \to \mathbb{R}^m$ is diff. at $p \in U$, then f is cts. at p.

– The matrix of the linear transformation Df_p is an $m \times n$ matrix, denoted $f'(p)$ & called the Jacobian matrix of f at p.

Thm: If $f: U \to \mathbb{R}^m$ is diff. at $p \in U$, then all the directional derivatives of f at p exist &

$$f'(p, v) = Df_p(v)$$

Cor: Let $\gamma : \mathbb{R} \to \mathbb{R}^m$ be a diff. path.
Then $\gamma'(x) = D\gamma_x(e_1)$

Pf: $D\gamma_x(e_1) = \gamma'(x, e_1) = \lim\limits_{t \to 0} \dfrac{\gamma(x + te_1) - \gamma(x)}{t}$

$$= \lim_{t \to 0} \frac{\gamma(x+t) - \gamma(x)}{t} = \gamma'(x)$$

Thm: If $f: U \to \mathbb{R}^m$ is diff at $p \in U$, then $\frac{\partial f_i}{\partial x_j}(p)$ exists for each $1 \le i \le m$, $1 \le j \le n$.

& $f'(p)$ is given by

$$f'(p) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(p) & \cdots & \frac{\partial f_1}{\partial x_n}(p) \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1}(p) & \cdots & \frac{\partial f_m}{\partial x_n}(p) \end{bmatrix}$$

Note: The converse of this thm is false. But if we add one additional cond^n, then it holds.

Thm: Let $f: U \to \mathbb{R}^m$. Suppose that all $\frac{\partial f_i}{\partial x_j}(p)$ exist for x in an open set containing p, & suppose that each $\frac{\partial f_i}{\partial x_j}$ is cts. at p. Then f is diff. at p.

Thm: (Chain Rule)

If $f: \mathbb{R}^n \to \mathbb{R}^m$ is diff. at $a \in \mathbb{R}^n$ & $g: \mathbb{R}^m \to \mathbb{R}^p$ is diff. at $b = f(a) \in \mathbb{R}^m$, then $g \circ f: \mathbb{R}^n \to \mathbb{R}^p$ is diff. at a &

$$D(g \circ f)_a = (Dg)_{f(a)} \cdot Df_a$$

Ppn: Let $\gamma: \mathbb{R} \to \mathbb{R}^m$ be a diff. path in $\mathbb{R}^m$.
Let $f: \mathbb{R}^m \to \mathbb{R}^p$. Let $\gamma(0) = p$ & $\gamma'(0) = v$.
Then $Df_p(v)$ is the initial tangent vector (also called velocity vec.) of $f \circ \gamma$, i.e.

$$(f \circ \gamma)'(0) = Df_p(v)$$

Pf: Consider the image path $f \circ \gamma$ in $\mathbb{R}^n$.
Set $a = 0$ & $b = \gamma(0)$.
Then the chain rule implies that $D(f \circ \gamma)_0 = Df_{\gamma(0)} \cdot D\gamma_0$

T'fore, $\forall\, w \in \mathbb{R}$, we obtain $D(f \circ \gamma)_0(w) = Df_{\gamma(0)} \cdot D\gamma_0(w)$
If we choose $w = e_1 \in \mathbb{R}$, then

LHS: $D(f \circ \gamma)_0(e_1) = (f \circ \gamma)'(0)$

RHS: $Df_{\gamma(0)} \cdot (D\gamma_0(e_1)) = Df_{\gamma(0)}(\gamma'(0)) = Df_p(v)$

Ppⁿ: Let $U \subseteq \mathbb{R}^n$ open. Let $f: U \to \mathbb{R}^n$ s.t $f: U \to f(U)$ is invertible. Suppose f is diff. at $x \in U$ & f^{-1} is diff. at $f(x)$. Then

$$(Df^{-1})_{f(x)} \cdot Df_x = Id$$

In particular, Df_x is an inv'ble linear transformation.

Thm: (Inverse fnⁿ thm)

Let $U \subseteq \mathbb{R}^n$ be open. Let $f: U \to \mathbb{R}^n$ be a fnⁿ of class C^r. Suppose Df_a is inv'ble at some $a \in U$. Then $\exists$ a (possibly smaller) nbd V of a & a nbd W of $f(a)$ s.t $f: V \to W$ is one-one & onto, and s.t the inverse fnⁿ $f^{-1}: W \to V$ is of class C^r.

Thm: Let $G \subseteq GL(n, \mathbb{K})$ be a mlg, with Lie alg. $\mathfrak{g} \subset \mathfrak{gl}(n, \mathbb{K})$.
Then $\forall X \in \mathfrak{g}$, $e^X \in G$.

Pf: Step 1: Let $\{X_1, \dots, X_k\}$ be a basis for $\mathfrak{g}$.
For each $i = 1, \dots, k$, let $\alpha_i : (-\epsilon, \epsilon) \to G$ be a diff. path with $\alpha_i(0) = I$ & $\alpha_i'(0) = X_i$

Let $\mathcal{U} = \{ t_1 X_1 + \dots + t_k X_k : t_i \in (-\epsilon, \epsilon) \}$
Then $\mathcal{U}$ is an open nbd of 0 in $\mathfrak{g}$.

Def. $F_g : \mathcal{U} \to G$ as follows:

$$F_g(c_1 X_1 + \dots + c_k X_k) = \alpha_1(c_1) \cdot \alpha_2(c_2) \cdot \dots \cdot \alpha_k(c_k)$$

Then, note that F_g has these properties:

1. $F_g(0) = \alpha_1(0) \cdot \alpha_2(0) \cdot \dots \cdot \alpha_k(0) = I \cdot I \cdot \dots \cdot I = I$
2. For each $i = 1, \dots, k$,

$$(DF_g)_0(X_i) = F_g'(0, X_i) = \lim_{t \to 0} \frac{F_g(0 + tX_i) - F_g(0)}{t}$$

$$= \lim_{t \to 0} \frac{\alpha_i(t) - \alpha_i(0)}{t} = \alpha_i'(0) = X_i$$

Thus, $(Df_g)_0(X_i) = X_i \quad \forall\ i = 1, \dots, k$

This implies that $(Df_g)_0(X) = X \quad \forall\ X \in g$

Step 2 : Choose a subsp. $p \in M_n(\mathbb{K})$ which is complementary to g, so we have $M_n(\mathbb{K}) = g \oplus p$.

Def. $f_p : p \to M_n(\mathbb{K})$ by

$$f_p(V) = I + V \quad \forall\ V \in p$$

Then $f_p(0) = I$ & $(Df_p)_0(Y) = Y \quad \forall\ Y \in p$

Step 3 : Let $B_\epsilon = \{ W \in M_n(\mathbb{K}) : \|W\| < \epsilon \}$

Def. $f : B_\epsilon \to M_n(\mathbb{K})$ by

$$f(X+Y) = f_g(X) \cdot f_p(Y) \quad \forall\ X \in g \ \&\ Y \in p.$$

Then

1. $f(0) = f_g(0) \cdot f_p(0) = I \cdot I = I$

2. For $X \in g$,

$$(Df)_0(X) = f'(0, X) = \lim_{t \to 0} \frac{f(0+tX) - f(0)}{t}$$

$$= \lim_{t \to 0} \frac{f_g(tX) - f_g(0)}{t}$$

$$= f_g'(0, X) = X \quad \text{(by Step 1)}$$

for $Y \in p$,

$$(DF)_0(Y) = F'(0, Y) = \lim_{t \to 0} \frac{F(0+tY) - F(0)}{t}$$

$$= \lim_{t \to 0} \frac{f_p(tY) - f_p(0)}{t}$$

$$= f_p'(0, Y) = Y \quad \text{(by Step 2)}$$

Hence, $(DF)_0(X+Y) = X+Y \quad \forall X \in g, \forall Y \in p$.

ie $(DF)_0$ is the identity linear transformation.

<u>Step 4</u>: By above step 3, $F : B_\epsilon \to M_n(\mathbb{K})$ satisfies

$F(0) = I$ & $(DF)_0$ is inv'ble

By the Inverse fxn thm, $\exists$ a nbd U_0 of 0 in B_ϵ & a nbd U_I of I in $M_n(\mathbb{K})$ s.t

$F : U_0 \to U_I$ is inv'ble.

Write the inv. fxn $F^{-1} : U_I \to U_0$ as follows:

$$F^{-1}(a) = \underbrace{u(a)}_{\in g} + \underbrace{v(a)}_{\in p} \quad \text{for each } a \in U_I$$

Then by def^n, $\forall\ X+Y \in B_\varepsilon$ with $X \in \mathfrak{g}$ & $Y \in \mathfrak{p}$, we have

$$u(F(X+Y)) = X \quad \& \quad v(F(X+Y)) = Y$$

Thus, v has this useful ppt.,
For $a \in \mathcal{U}_I$ (nbd of I in $M_n(\mathbb{K})$), we have $v(a)=0 \Rightarrow a \in G$.

Pf: $a = F(F^{-1}(a)) = F(u(a)+v(a))$

If $v(a)=0$, $\quad a = F(u(a)) = F_{\mathfrak{g}}(u(a)) \in G$

Step 5: Claim: $\exists$ a nbd $\tilde{\mathcal{U}}_I \ni I$ s.t $\forall\ a \in \tilde{\mathcal{U}}_I$ & $\forall\ X \in \mathfrak{g}$
$(Dv)_a(X \cdot a) = 0$

Pf: Write a as:
$$a = F(Z+Y) = F_{\mathfrak{g}}(Z) \cdot F_{\mathfrak{p}}(Y) \quad \text{where } Z \in \mathfrak{g},\ Y \in \mathfrak{p}$$

For each elem $W \in \mathfrak{g}$ & $\forall\ t$ small enough,
$Z + tW \in \mathcal{U}$ = domain of $F_{\mathfrak{g}}$ &
$$Y = v(F(\underbrace{Z+tW}_{\in \mathfrak{g}} + \underbrace{Y}_{\in \mathfrak{p}}))$$

$$= v(f_g(Z+tW)\cdot f_p(Y))$$

That is, RHS is const. in t.

Hence, $$0 = \frac{d}{dt}\Big|_{t=0} v(f_g(Z+tW)\cdot f_p(Y))$$
$$= Dv_a((Df_g)_Z(W)\cdot f_p(Y))$$
$$= Dv_a(\underbrace{(Df_g)_Z(W)\cdot f_g^{-1}(Z)}_{X}\cdot a)$$
$$= Dv_a(X\cdot a)$$

where we have written
$$X = (Df_g)_Z(W)\cdot f_g^{-1}(Z) \quad -(*)$$

If we can show that every $X\in\mathfrak{g}$ is of the form $(*)$ for some $W\in\mathfrak{g}$, that would complete the proof of the claim.

To show this, first observe that if $\gamma(t)$ is the path in G given by $\gamma(t) = f_g(Z+tW)\cdot f_g(Z)^{-1}$

Then, $\gamma'(0) = (Df_g)_Z(W)\cdot f_g(Z)^{-1}$

So, by def^n of $\mathfrak{g}$, $(DF_{\mathfrak{g}})_z(W) \cdot F_{\mathfrak{g}}^{-1}(z)$ is an element of $\mathfrak{g}$.

This observation gives us a map,

$$\begin{aligned} \Psi_z : \mathfrak{g} &\to \mathfrak{g} \\ W &\mapsto (DF_{\mathfrak{g}})_z(W) \cdot F_{\mathfrak{g}}(z)^{-1} \end{aligned}$$

If $z=0$, $\Psi_z = \Psi_0$ is the identity map
(hence has $\det(\Psi_0) = 1$)

So, by continuity, if z is close to 0, then $\det(\Psi_z)$
is close to $\det(\Psi_0) = 1$.
So, Ψ_z is inv'ble. Hence $\forall X \in \mathfrak{g}$, $\exists W \in \mathfrak{g}$ s.t

$$(DF_{\mathfrak{g}})_z(W) \cdot F_{\mathfrak{g}}(z)^{-1} = X$$

That completes the proof of the claim.

Step 6: Let $X \in \mathfrak{g}$. Def. $a(t) = e^{tX}$.

We want to prove that $a(t) \in G$ for small t.

By Step 4, it is enough to show that $v(a(t)) = 0$

Note that, $v(a(0)) = v(I) = 0$

T'fore, it is enough to show that, $\frac{d}{dt}(v(a(t)) = 0$

for small t.

We have,

$$\frac{d}{dt} v(a(t)) = (Dv)_{a(t)} \cdot a'(t)$$
$$= (Dv)_{a(t)} (X \cdot a(t))$$
$$= 0$$

This allows us to conclude that for small t, $e^{tX} \in G$

(say for $t \in (-\epsilon_0, \epsilon_0)$)

Then for any $N \in \mathbb{Z}_{>0}$, $e^{NtX} = e^{tX + \dots + tX}$

$$= e^{tX} \cdot e^{tX} \dots e^{tX} \in G$$

This proves that $e^{xX} \in G$, $\forall x \in \mathbb{R}$.

This completes the proof.

Pp^n: $\exp : M_n(\mathbb{K}) \to M_n(\mathbb{K})$ is a C^∞ fn^n.

Lem: $\exists\, r > 0$ s.t $V = \exp(\underset{\text{Ball of radius } r \text{ around } 0}{B_r})$ is a nbd of I in $GL(n, \mathbb{K})$
& $\exp : B_r \to V$ is a homeomorphism
(which is C^∞ & has C^∞ inverse)

Pf: Let $X \in M_n(\mathbb{K})$.

Then $$(D(\exp))_0(X) = \frac{d}{dt}\Big|_{t=0} (e^{tX}) = X$$

i.e $(D(\exp))_0$ is the identity map (identity linear transformation),
i.e., it is inv'ble.

By Inverse fn^n thm, $\exists$ a nbd U_1 of 0 in $M_n(\mathbb{K})$ &
$\exists$ a nbd of U_2 of I in $M_n(\mathbb{K})$ s.t.
$\exp : U_1 \to U_2$ is a C^∞ diffeomorphism.

Since $\forall\, X \in M_n(\mathbb{K})$, $e^X \in GL(n, \mathbb{K})$ & because $GL(n, \mathbb{K}) \subseteq M_n(\mathbb{K})$
this completes the proof.

Rem: The inverse of exp is denoted 'log', it is a
C^∞ fn^n defined on nbd of I.

Coming up later: An analogous statement for mlg $G \subseteq GL(n, \mathbb{K})$

Thm: Let $G \subseteq GL(n, \mathbb{K})$ be a mlg with Lie alg. $\mathfrak{g} \subseteq \mathfrak{gl}(n, \mathbb{K})$. Then $\exists\, r > 0$ s.t $V = \exp(B_r \cap \mathfrak{g})$ is a nbd of I in G & the restriction $\exp : B_r \cap \mathfrak{g} \to V$ is a homeomp.

Lem: Let $G \subseteq GL(n, \mathbb{K})$ be a mlg with Lie alg. $\mathfrak{g} \subseteq \mathfrak{gl}(n, \mathbb{K})$. In the prev. lemma, $r > 0$ can be chosen s.t additionally,
$$\exp(B_r \cap \mathfrak{g}) = \exp(B_r) \cap G$$

Pf: For any r, the inclusions $\underbrace{\exp(B_r \cap \mathfrak{g}) \subseteq \exp(B_r)}_{\text{(Trivial)}}$ & $\underbrace{\exp(B_r \cap \mathfrak{g}) \subseteq G}_{\text{(by Thm)}}$ hold

Hence, $\exp(B_r \cap \mathfrak{g}) \subseteq \exp(B_r) \cap G$

Claim: For r sufficiently small, the reverse inclusion $\exp(B_r) \cap G \subseteq \exp(B_r \cap \mathfrak{g})$ also holds

Before proving this claim, some preparation.

Choose subsp. $\mathfrak{p} \subset M_n(\mathbb{K})$ complementary to $\mathfrak{g}$, s.t

$$M_n(\mathbb{K}) = \mathfrak{g} \oplus \mathfrak{p}$$

Def. $\varphi : M_n(\mathbb{K}) \to M_n(\mathbb{K})$ by

$$\varphi(X+Y) = e^X \cdot e^Y \quad \text{for } X \in \mathfrak{g},\ Y \in \mathfrak{p}.$$

Then $\varphi|_{\mathfrak{g}} = \exp|_{\mathfrak{g}}$

Also, $(D\varphi)_0$ is the identity map, hence inv'ble.

Hence, the Inverse fxn thm implies that φ is locally inv'ble.

Now, back to the proof of the claim.

Suppose the claim is false.

Then $\exists$ a seq. of non-zero elems. $\{B_1, B_2, \dots\}$ in $M_n(\mathbb{K})$ st

$\|B_i\| \to 0$, $\exp(B_i) \in G$ $\forall\, i$, but $B_i \notin \mathfrak{g}$.

We will actually use a related statement.

Assuming the claim is false, we can obtain a seq. of non-zero elems. $\{A_1, A_2, \dots\}$ in $M_n(\mathbb{K})$ s.t $\|A_i\| \to 0$,

$\varphi(A_i) \in G$ $\forall\, i$, but $A_i \notin \mathfrak{g}$

We can write : $A_i = X_i + Y_i$, where $X_i \in \mathfrak{g}$, $Y_i \in \mathfrak{p}$, $Y_i \neq 0$

Def: $g_i = \varphi(A_i) = e^{X_i} \cdot e^{Y_i} \in G$

Then $e^{Y_i} = e^{-X_i} \cdot g_i \in G$

Now, $\left\{ \frac{Y_1}{\|Y_1\|}, \frac{Y_2}{\|Y_2\|}, \ldots \right\}$ is a seq. of unit vecs. in $\mathfrak{p}$, i.e it is a seq. in the unit sphere S inside $\mathfrak{p}$. S is a compact set.

Hence, the seq. $\{ Y_1/\|Y_1\|, Y_2/\|Y_2\|, \ldots \}$ must have a conv. subseq., conv. to some vec. Y in $\mathfrak{p}$.

$$Y_{n_k}/\|Y_{n_k}\| \to Y$$

Now, let $t \in \mathbb{R}$. Since $Y_{n_k}/\|Y_{n_k}\| \to Y$ & $\|Y_{n_k}\| \to 0$, it is possible to choose a seq. of integers N_k s.t $N_k Y_{n_k} \to tY_{n_k}$.

Then, $$e^{N_k Y_{n_k}} = \left(e^{Y_{n_k}}\right)^{N_k} \in G \quad \forall k$$

Since G is closed in $GL(n, \mathbb{K})$, this implies that $e^{tY} \in G$

This means that $e^{tY} \in G \ \forall t \in \mathbb{R}$

So, differentiating the path $\gamma(t) = e^{tY}$ at $t=0$,

we obtain $Y \in \mathfrak{g}$.

But $Y \in \mathfrak{p}$ & $Y \neq 0$. $\rightarrow$ Contd^n.

Note: In the last lemma, the assumption that G is a mlg is important.

eg: Let $a \in \mathbb{Q}^c$

Let $G = \left\{ \begin{bmatrix} e^{it} & 0 \\ 0 & e^{ita} \end{bmatrix} : t \in \mathbb{R} \right\} \subseteq GL(2, \mathbb{C})$

The Lie alg. of G is

$$\mathfrak{g} = \operatorname{span}_{\mathbb{R}} \left\{ \underbrace{\begin{bmatrix} i & 0 \\ 0 & ia \end{bmatrix}}_{W} \right\}$$

Note that $e^{tW} = \begin{bmatrix} e^{it} & 0 \\ 0 & e^{ita} \end{bmatrix}$

Then $B_r \cap \mathfrak{g} = \left\{ tW : t \in \left(\frac{-r}{\sqrt{1+a^2}}, \frac{r}{\sqrt{1+a^2}} \right) \right\}$

& $\exp(B_\varepsilon \cap \mathfrak{g}) = \left\{ \begin{bmatrix} e^{it} & 0 \\ 0 & e^{ita} \end{bmatrix} : t \in \left(\frac{-\varepsilon}{\sqrt{1+a^2}}, \frac{\varepsilon}{\sqrt{1+a^2}} \right) \right\}$

But, $\exp(B_\varepsilon \cap \mathfrak{g})$ is <u>not</u> a nbd of I in G.

Any nbd of I in G contains pts. of the form $e^{2\pi i n w}$ for n arbitrarily large.

But for large n, $e^{2\pi i n w} \notin \exp(B_\varepsilon \cap \mathfrak{g})$

Thm: Let $G \subset GL(n, \mathbb{K})$ be a mlg with Lie algebra $\mathfrak{g} \subseteq \mathfrak{gl}(n, \mathbb{K})$.
Then $\exists\, r > 0$ s.t. $V = \exp(B_r \cap \mathfrak{g})$ is a nbd of I in G,
& the restriction $\exp : B_r \cap \mathfrak{g} \to V$ is a homeomp.

Pf: Let $r > 0$ as in the prev. lemma. Then

$V = \exp(B_r \cap \mathfrak{g})$ is a nbd of I in G

(since prev. lemma implies that $V = \exp(B_r) \cap G$,
which further implies that $\exp(B_r)$ is open in $M_n(\mathbb{K})$)

The restriction

$$\exp : B_r \cap \mathfrak{g} \to V \text{ is cts.}$$

The inverse fn from V to $B_r \cap \mathfrak{g}$ is cts.
(restriction of V to the cts. fn by $\log : \exp(B_r) \to B_r$)

Note: In this proof, $\exp : B_r \cap \mathfrak{g} \to V$
is not just cts. but is smooth (C^∞). Its inverse
$\log : V \to B_r \cap \mathfrak{g}$ is the restriction to V of the smooth fn
log.

Defn: If $X \subset \mathbb{R}^m$ (any set), then $f: X \to \mathbb{R}^n$ is said to be **smooth** if $\forall\, p \in X$, $\exists$ a nbd U of p in $\mathbb{R}^m$ & a smooth (C^∞) fxn $\tilde{f}: U \to \mathbb{R}^n$ s.t $\tilde{f}|_{X \cap U} = f$

Defn: $X \subset \mathbb{R}^{m_1}$ & $Y \subseteq \mathbb{R}^{m_2}$ are said to be **diffeomorphic** if $\exists$ a bijective fxn $f: X \to Y$ s.t f is smooth & s.t $f^{-1}: Y \to X$ is also smooth. In this case, f is said to be a **diffeomorphism**.

Rem: In the previous thm, the word homeomp. can be replaced by diffeomorphism

Note: A diffeomorphism is a homeomp., which is smooth & has smooth inverse.

Defn: A subset $X \subseteq \mathbb{R}^N$ is called a manifold of dimension n if for all $p \in X$, $\exists$ a nbd V of p in X which is diffeomp. to an open set $U \subseteq \mathbb{R}^n$.

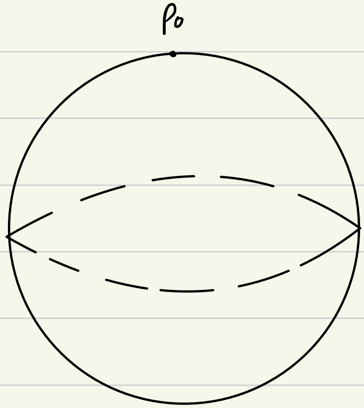
Note: To rigorously prove that a set X is a manifold, you must construct a parametrization at each $p \in X$, ie a diffeomp. φ from an open set $U \subseteq \mathbb{R}^n$ to a nbd V of p in X.

eg: $S^2 = \{ (x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2=1 \}$

Pp^n : $S^2 \subset \mathbb{R}^3$ is a 2-dimensional manifold

Pf: Consider $p_0 = (0,0,1) \in S^2$

$V = \{ (x,y,z) \in S^2 : z > 0 \}$ is a nbd of p in S.



Let $U = \{ (x,y) \in \mathbb{R}^2 : x^2+y^2 < 1 \}$

be an open set in $\mathbb{R}^2$.

Def. $\varphi : U \to V$ as

$\varphi(x,y) = (x, y, \sqrt{1-x^2-y^2})$

Then φ is smooth & bijective &

$\varphi^{-1} : V \to U$ is given by

$\varphi^{-1}(x,y,z) = (x,y)$

Note that φ^{-1} is smooth since it extends to the smooth

fx^n $\psi : \mathbb{R}^3 \to \mathbb{R}^2$ given by

$\psi(x,y,z) = (x,y)$

for arbitrary $p \in S^1$, if $A \in SO(3)$ is any matrix with $T_A(p_0) = p$

Then $T_A \circ \varphi : \mathcal{U} \to T_A(V)$ is a parametrization at p.

<u>Thm</u>: Any mlg G of dim n is a manifold of dim n.

<u>Pf</u>: Let $G \subset GL(n, \mathbb{K})$ is a mlg of dim n, with Lie alg. $\mathfrak{g}$

Then $\exists\ r > 0$ s.t $V = \exp(B_r \cap \mathfrak{g})$ is a nbd of I in G. $(\simeq \mathbb{R}^n)$

Then, the restriction $\exp : B_r \cap \mathfrak{g} \to V$ is a parametrization at I.

Now, consider an arbitrary pt. $g \in G$. We want to find a parametrization at g.

Def. $L_g : M_n(\mathbb{K}) \longrightarrow M_n(\mathbb{K})$

$$L_g(A) = gA$$

Then :

1. $L_g : M_n(\mathbb{K}) \to M_n(\mathbb{K})$ is a diffeomp.
   (note that L_g is linear with inverse $L_{g^{-1}}$)

2. L_g restricts to a map $L_g : G \to G$

T'fore, $L_g : G \to G$ is a diffeomp.

$\Rightarrow$ $L_g(V)$ is a nbd of g in G, and

$L_g \circ \exp : B_\varepsilon \cap \mathfrak{g} \to L_g(V)$ is a parametrization at g.

<u>Recall</u>: for $X \subseteq \mathbb{R}^n$ & $p \in X$, the tangent sp. to X at p is def. as

$$T_p X = \{ \gamma'(0) : \gamma : (-\epsilon, \epsilon) \to X \text{ is a diff. path with } \gamma(0) = p \}$$

<u>Ppn</u>: If $X \subset \mathbb{R}^m$ is an n-dim manifold, then $\forall\, p \in X$,

$T_p X$ is an n-dim subsp. of $\mathbb{R}^m$.

<u>Pf</u>: Let $\underset{\subseteq \mathbb{R}^n}{\varphi : U} \to \underset{\subset X}{V}$ be a parametrization at p

Wlog we can assume that $0 \in U$ & $\varphi(0) = p$

<u>Claim</u>: $T_p X = D\varphi_0(\mathbb{R}^n)$

<u>Pf</u>: Let $\psi : V \to U$ be the inverse of φ.

Let $\tilde{V} \subseteq \mathbb{R}^m$ be an open set with $V \subset \tilde{V}$ & let $\tilde{\psi} : \tilde{V} \to \mathbb{R}^n$

be a smooth fn s.t $\tilde{\Psi}|_V = \Psi$

Now, let $v \in T_pX$ i.e $\gamma : (-\epsilon, \epsilon) \to V \subset X$ be a diff. path with $\gamma(0) = p$, $\gamma'(0) = v$.

Then, $\alpha = \Psi \circ \gamma = \tilde{\Psi} \circ \gamma$ is a diff. path in U with $\alpha(0) = 0 \in \mathbb{R}^n$; & $\varphi(\alpha) = \varphi(\Psi \circ \gamma) = id_V \circ \gamma = \gamma$

So, $D\varphi_0(\alpha'(0)) = \gamma'(0)$.

This shows $T_pX \subset D\varphi_0(\mathbb{R}^n)$

On the other hand, if $w \in \mathbb{R}^n$, then $\beta(t) = tw$ is a diff. path in U with $\beta(0) = 0$ & $\beta'(0) = w$.

Then, $D\varphi_0(w) = (\varphi \circ \beta)'(0) \in T_pX$, since $(\varphi \circ \beta)$ is a diff. path in X with $(\varphi \circ \beta)(0) = p$

This shows that $D\varphi_0(\mathbb{R}^n) \subseteq T_pX$

This proves the claim.

Now, observe that $\tilde{\psi} \circ \varphi = id_{\mathcal{U}}$, so this implies that $D\tilde{\psi}_p \circ D\varphi_0 = I_{\mathbb{R}^n}$. This implies $D\varphi_0$ has rank n.

Then, since $T_pX = D\varphi_0(\mathbb{R}^n)$, this shows that T_pX is an n-dim subsp. of $\mathbb{R}^m$.

<u>Def^n</u>: Let $X_1 \subset \mathbb{R}^{m_1}$ & $X_2 \subset \mathbb{R}^{m_2}$ be manifolds.
Let $f: X_1 \to X_2$ be a smooth fn^n. Let $p \in X_1$. Let $q = f(p) \in X_2$.
Since f is smooth, $\exists$ open set $W \subset \mathbb{R}^{m_1}$ s.t $p \in W$ & a smooth fn^n $F: W \to \mathbb{R}^{m_2}$ s.t $F\big|_{W \cap X_1} = f$.
If $v \in T_pX$, we define $Df_p(v)$ be to be equal to $DF_p(v)$.

<u>Pp^n</u>: In the above def^n, the map $v \mapsto Df_p(v)$ is a well-defined linear map from T_pX_1 to $T_{f(p)}X_2$.

<u>Pf</u>: We need to show that Df_p does <u>not</u> depend on the choice of F.
Choose parametrizations $\varphi_1 : \underset{\subseteq \mathbb{R}^{n_1}}{U_1} \to \underset{\subseteq X_1}{V_1}$ with $\varphi_1(0) = p \in X_1$

$\varphi_2 : \underset{\subseteq \mathbb{R}^{n_2}}{U_2} \to \underset{\subseteq X_2}{V_2}$ with $\varphi_2(0) = f(p) \in X_2$

If needed, we can replace U_1 with a smaller open set s.t $\varphi_1(U_1) \subset W$ & s.t f maps $\varphi_1(U_1)$ into $\varphi_2(U_2)$.

Hence, $h = \varphi_2^{-1} \circ f \circ \varphi_1 : U_1 \to U_2$
is well-defined, & is a smooth map.

We have the following diagram of maps which satisfies

$$f \circ \varphi_1 = \varphi_2 \circ h$$

$$\begin{tikzcd} W \arrow[r, "f"] & \mathbb{R}^{m_2} \\ U_1 \arrow[u, "\varphi_1"] \arrow[r, "h"] & U_2 \arrow[u, "\varphi_2"'] \end{tikzcd}$$

Taking derivatives, we have

$$\begin{tikzcd} \mathbb{R}^{m_1} \arrow[r, "Df_p"] & \mathbb{R}^{m_2} \\ \mathbb{R}^{n_1} \arrow[u, "(D\varphi_1)_0"] \arrow[r, "Dh_0"] & \mathbb{R}^{n_2} \arrow[u, "(D\varphi_2)_0"'] \end{tikzcd}$$

The chain rule implies that

$$Df_p \circ (D\varphi_1)_0 = (D\varphi_2)_0 \circ Dh_0 \quad - (*)$$

Proof of the prev. ppⁿ $\Rightarrow$ Image$((D\varphi_1)_0) = T_p X_1$

& Image$((D\varphi_2)_0) = T_{f(p)} X_2$

T'fore, $(*)$ tells us that Df_p maps $T_p X_1$ into $T_{f(p)} X_2$.

We have

$$\begin{tikzcd} T_p X_1 \arrow[r, "Df_p"] & T_{f(p)} X_2 \\ \mathbb{R}^{n_1} \arrow[u, "(D\varphi_1)_0"] \arrow[r, "Dh_0"] & \mathbb{R}^{n_2} \arrow[u, "(D\varphi_2)_0"'] \end{tikzcd}$$

In this diagram, the vertical maps are isomorphisms.

This implies that $Df_p : T_p X_1 \to T_{f(p)} X_2$ equals the map

$$(D\varphi_2)_0 \circ Dh_0 \circ (D\varphi_1)_0^{-1} : T_p X_1 \to T_{f(p)} X_2$$

This tells us that Df_p is well-defined

(independent of choice of F).

Ppn : Let $f : X_1 \to X_2$ be a smooth fnn b/w manifolds & let $p \in X_1$.
If $v \in T_p X_1$, then $Df_p(v) \in T_{f(p)} X_2$ equals $(f \circ \gamma)'(0)$ for any diff. path $\gamma(t)$ in X_1 with $\gamma(0) = p$ & $\gamma'(0) = v$.

Pf: Let $W \subset \mathbb{R}^{m_1}$ be an open set with $p \in W$ &
$F : W \to \mathbb{R}^{m_2}$ smooth fnn s.t $F|_{X_1 \cap W} = f$.
Then by defn, $Df_p(v) = DF_p(v)$

But, $DF_p(v) = (F \circ \gamma)'(0)$ (for any diff. path $\gamma(t)$ in W with $\gamma(0) = p$ & $\gamma'(0) = v$)

If $\gamma(t) \subseteq X_1 \ \forall \ t$, $F \circ \gamma = f \circ \gamma$

So, $Df_p = (f \circ \gamma)'(0)$

Ppn: (Chain Rule for manifolds)

Suppose $f: X_1 \to X_2$ & $g: X_2 \to X_3$ are smooth fxn b/w manifolds. Then $g \circ f: X_1 \to X_3$ is smooth as well, and, $\forall\, p \in X_1$,

$$D(g \circ f)_p = Dg_{f(p)} \circ Df_p$$

Thm: (Inverse fxn thm for manifolds)

Let $f: X_1 \to X_2$ be a smooth fxn b/w manifolds. Let $p \in X_1$. Suppose Df_p is invertible. Then, $\exists$ a nbd U of p in X_1, & a nbd V of $f(p)$ in X_2 s.t $f: U \to V$ is a diffeomp.

Thm: Let G be a mLg. Let $\varphi: (\mathbb{R}, +) \to G$ be a cts. gp. homomp. Then φ is smooth (C^∞).

Pf: Note, that
$$\begin{aligned} \varphi(t) &= \varphi(x + t - x) \\ &= \varphi(x) \cdot \varphi(t - x) \\ &= L_{\varphi(x)} \circ \varphi \circ T_{-x}(t) \end{aligned}$$

where $T_{-x}: \mathbb{R} \to \mathbb{R}$ is defined by
$$t \mapsto t - x$$

Hence, $\varphi\Big|_{(x-\epsilon,\, x+\epsilon)} = L_{\varphi(x)} \circ \varphi\Big|_{(-\epsilon,\epsilon)} \circ T_{-x}$

T'fore, it is enough to prove that φ is C^∞ on a nbd of 0 in $\mathbb{R}$.

Now, to prove that φ is smooth on a nbd of 0.
Let $r>0$ s.t $\exp : B_r \cap \mathfrak{g} \to V$ is a diffeomp.

Let $U := B_r \cap \mathfrak{g}$ & let $U_1 = B_{r/2} \cap \mathfrak{g}$
Let $\epsilon > 0$ s.t $\varphi(t) \in \exp(U_1)$ $\forall$ $|t| < \epsilon$

Let $n \in \mathbb{Z}_{>0}$. Then there are uniquely determined $X, Y \in U_1$ s.t
$e^X = \varphi(\epsilon/n)$ & $e^Y = \varphi(\epsilon)$

Claim : $nX = Y$
Pf : Since $e^{nX} = e^X \dots e^X = \varphi(\epsilon/n) \dots \varphi(\epsilon/n) = \varphi(\epsilon) = e^Y$
& since exp is inj on U_1, it is enough to show that $nX \in U_1$.

Now, $X \in U_1$, so $1 \leq j \leq n$ & assume that $jX \in U_1$.
We will prove that $(j+1)X \in U_1$

First, note that $(j+1)X = \left(\frac{j+1}{j}\right) jX \in U_1$ & $\exp((j+1)X) = \varphi((j+1)\epsilon/n)$
$\in \exp(U_1)$

Since exp is inj. on U (& hence on U_1), it follows that
$(j+1)X \in U_1$

We conclude that $nX \in U_1$, so $nX = Y$ as claimed.

Now, let $m \in \mathbb{Z}$ with $0 < |m| \leq n$.

If $m > 0$, then $\varphi(m\epsilon/n) = \varphi(\epsilon/n)^m = \exp(Y/n)^m = \exp\left(\frac{m}{n}Y\right)$

If $m < 0$, then $\varphi\left(\frac{m\epsilon}{n}\right) = \varphi\left(-\frac{m\epsilon}{n}\right)^{-1} = \exp\left(-\frac{mY}{n}\right)^{-1} = \exp\left(\frac{mY}{n}\right)$

Thus, for all $s \in [-1, 1] \cap \mathbb{Q}$,

$$\varphi(s \cdot \epsilon) = \exp(sY)$$

By continuity, this implies that $\varphi(s \cdot \epsilon) = \exp(sY) \quad \forall\, s \in [-1, 1]$

$$\Rightarrow \varphi(t) = \exp\left(t \cdot \frac{Y}{\epsilon}\right) \quad \forall\, t \in [-\epsilon, \epsilon]$$

Hence, φ is C^∞.

Thm: If H & G are mlg & let $\Phi : H \to G$ be a cts. homomp. (Lie gp. homomp.). Then Φ is C^∞.

Pf: Suppose H is of dimension d.
Let $\{X_1, \dots, X_d\}$ be a basis of $\mathfrak{h}$ (Lie algebra of H)

The map $\alpha : \mathbb{R}^d \to H$ defined by
$$\alpha(t_1, t_2, \dots, t_d) = (e^{t_1 X_1}) \dots (e^{t_d X_d}) \quad \text{is } C^\infty,$$

& satisfies $D\alpha_0(e_i) = X_i$ for each $1 \leq i \leq d$.
(Here $e_i = (0, \dots, \underset{i\text{th position}}{1}, \dots, 0) \in \mathbb{R}^d$)

So, $D\alpha_0$ is non-singular. By inverse fn thm, there is a nbd U of 0 in $\mathbb{R}^d$ & a nbd V of I in H s.t $\alpha : U \to V$ is a diffeomp.

Now, for each $1 \leq i \leq d$, $t \mapsto \Phi(\exp(tX_i))$ is a cts. homomp. of $\mathbb{R}$ into G, hence, is C^∞ by the prev. thm.

Hence, $\Phi \circ \alpha$ is C^∞.
T'fore $\Phi|_V = (\Phi \circ \alpha) \circ \alpha^{-1}|_V$ is C^∞.

For any $\sigma \in H$, $v \in \sigma V$, we have

$$\Phi(v) = \Phi(\sigma)\,\Phi(\sigma^{-1}v) = L_{\Phi(\sigma)} \circ \Phi(L_{\sigma^{-1}}(v))$$

So, $\Phi|_{\sigma V} = L_{\Phi(\sigma)} \circ \Phi|_V \circ L_{\sigma^{-1}}|_{\sigma V}$

Hence, Φ is C^∞ on all of H.

Lie Bracket

Let G be a mlg, with Lie alg. $\mathfrak{g}$.

For each $g \in G$, we can define the conjugation map

$$\begin{aligned} C_g : G &\to G \\ a &\mapsto gag^{-1} \end{aligned}$$

This map C_g is a smooth isom.

The derivative $D(C_g)_I : \mathfrak{g} \to \mathfrak{g}$ is a vector sp. isom.

We denote this vec. sp. isom. by $\mathrm{Ad}_g := D(C_g)_I$

Ppn: For any $B \in \mathfrak{g}$, $Ad_g(B) = gBg^{-1}$

Pf: Let $b(t)$ be a diff. path in G with $b(0) = I$ with $b'(0) = B$.

Then,
$$Ad_g(B) = D(C_g)_I(B) = D(C_g)_I(b'(0))$$
$$= (C_g \circ b)'(0) = \frac{d}{dt}\Big|_{t=0} (g b(t) g^{-1})$$
$$= gBg^{-1}$$

Defn: The Lie Bracket of $A, B \in \mathfrak{g}$ is defined as

$$[A, B] := \frac{d}{dt}\Big|_{t=0} Ad_{a(t)}(B)$$

where $a(t)$ is any diff. path in G with $a(0) = I$ & $a'(0) = A$

Note: $[A, B] \in \mathfrak{g}$, since it is the initial velocity vector of a path in $\mathfrak{g}$.
(i.e $Ad_{a(t)}(B)$)

Ppn : For all $A, B \in \mathfrak{g}$, $[A, B] = AB - BA$

Pf : Let $a(t)$ be diff paths in G with
$a(0) = I$, $a'(0) = A$

Then,
$$[A, B] = \frac{d}{dt}\Big|_{t=0} \left(Ad_{a(t)}(B)\right)$$
$$= \frac{d}{dt}\Big|_{t=0} \left(a(t) B a(t)^{-1}\right)$$
$$= a'(0) B a(0) + a(0) B (-a'(0))$$
$$= AB - BA$$

$$\left\{ \begin{aligned} & a(t) a^{-1}(t) = I \\ \Rightarrow\ & a'(t) a^{-1}(t) + a(t)(a^{-1}(t))' = 0 \\ \Rightarrow\ & (a^{-1}(t))'\Big|_{t=0} = -a'(0) \end{aligned} \right\}$$

Note : $[A, B] = 0$ iff A & B commute

$\underline{\text{Ppn}}$: Let $A, B \in \mathfrak{g}$

$\underline{1.}$ If $[A, B] = 0$, then e^{tA} commutes with e^{sB} $\forall\ s, t \in \mathbb{R}$

$\underline{2.}$ If e^{tA} & e^{sB} commute for $t, s \in (-\epsilon, \epsilon)$, then $[A, B] = 0$

$\underline{\text{Pf}}$: $\underline{1.}$ If $[A, B] = 0$, then $AB = BA$, so

$$e^{tA} \cdot e^{sB} = e^{tA + sB} = e^{sB + tA} = e^{sB} \cdot e^{tA}$$

$\underline{2.}$ Fix $t \in (-\epsilon, \epsilon)$, then

$$\mathrm{Ad}_{(e^{tA})}(B) = e^{tA} B e^{-tA} = \left.\frac{d}{ds}\right|_{s=0} e^{tA} \cdot e^{sB} \cdot e^{-tA}$$

$$= \left.\frac{d}{ds}\right|_{s=0} e^{sB} \underbrace{e^{tA} \cdot e^{-tA}}_{I}$$

$$= \left.\frac{d}{ds}\right|_{s=0} e^{sB} = B$$

So, $\mathrm{Ad}_{(e^{tA})}(B) = B \quad \forall\ t \in (-\epsilon, \epsilon)$

$$\Rightarrow [A, B] = \left.\frac{d}{dt}\right|_{t=0} \left(\mathrm{Ad}_{(e^{tA})}(B)\right) = \left.\frac{d}{dt}\right|_{t=0} (B) = 0$$

$\underline{Pp^n}$: for all $A, A_1, A_2, B, B_1, B_2, C \in \mathfrak{g}$ & $\lambda_1, \lambda_2 \in \mathbb{R}$,

1. $[\lambda_1 A_1 + \lambda_2 A_2, B] = \lambda_1 [A_1, B] + \lambda_2 [A_2, B]$
2. $[A, \lambda_1 B_1 + \lambda_2 B_2] = \lambda_1 [A, B_1] + \lambda_2 [A, B_2]$
3. $[A, B] = -[B, A]$
4. (Jacobi identity)
   $[[A,B],C] + [[B,C],A] + [[C,A],B] = 0$

$\underline{Def^n}$: Let $\mathfrak{g}_1$ & $\mathfrak{g}_2$ be two Lie algs. A fx^n $f: \mathfrak{g}_1 \to \mathfrak{g}_2$ is called a Lie alg. homomp. if it satisfies both these cond^{n}s.

1. f is linear
2. $\forall A, B \in \mathfrak{g}_1$, $f([A,B]) = [f(A), f(B)]$

If f is also bijective, then f is called a Lie alg. isomp.

Ppn: Let G_1, G_2 be mlgs with Lie algs. $\mathfrak{g}_1$ & $\mathfrak{g}_2$ resp.
Let $f: G_1 \to G_2$ be a Lie gp. homomp. Then the derivative
$Df_I: \mathfrak{g}_1 \to \mathfrak{g}_2$ is a Lie alg. homomp.

Pf: Let $A, B \in \mathfrak{g}_1$. Let $a(t)$ & $b(t)$ be diff. paths in G.
with $a(0) = b(0) = I$, $a'(0) = A$, $b'(0) = B$.

Claim: For each $g \in G$, we have

$$Df_I(Ad_g(B)) = Ad_{f(g)}(Df_I(B))$$

Pf: Let $\gamma(t) = g\, b(t)\, g^{-1}$, then $\gamma(0) = I$ & $\gamma'(0) = Ad_g(B)$.

Hence, $$Df_I(Ad_g(B)) = Df_{\gamma(0)}(\gamma'(0)) = (f \circ \gamma)'(0)$$
$$= \left.\frac{d}{dt}\right|_{t=0} (f(g\, b(t)\, g^{-1})) = \left.\frac{d}{dt}\right|_{t=0} f(g)\, f(b(t))\, f(g)^{-1}$$
$$= Ad_{f(g)}((f \circ b)'(0))$$
$$= Ad_{f(g)}(Df_I(B))$$

Now, consider $Df_I : \mathfrak{g}_1 \to \mathfrak{g}_2$.

Since Df_I is linear, for any $w \in \mathfrak{g}_1$, $D(Df_I)_w = Df_I$

Now, let $v(t) = Ad_{a(t)}(B)$.

Then $v(t)$ is a path in $\mathfrak{g}_1$.

We have

$$Df_I(v'(0)) = D(Df_I)_{v(0)}(v'(0)) = \frac{d}{dt}\Big|_{t=0} (Df_I(v(t)))$$

Hence,
$$Df_I([A,B]) = Df_I\left(\frac{d}{dt}\Big|_{t=0} (Ad_{a(t)}(B))\right)$$
$$= \frac{d}{dt}\Big|_{t=0} \left(Df_I(Ad_{a(t)}(B))\right)$$
$$= \frac{d}{dt}\Big|_{t=0} \left(Ad_{f(a(t))}(Df_I(B))\right)$$
$$= [(f \circ a)'(0), Df_I(B)]$$
$$= [Df_I(A), Df_I(B)]$$

Rem: Lie bracket captures (infinitesimally) failure of elems in G to commute.

Cor: If $f: G_1 \to G_2$ is a Lie gp. isomp., then $\mathfrak{g}_1$ & $\mathfrak{g}_2$ are isomorphic.

Pf: By the ppn, $Df_I : \mathfrak{g}_1 \to \mathfrak{g}_2$ is a Lie alg. homomp.
And, since f is a diffeomp., Df_I is bij.
Hence, Df_I is a Lie alg. isomp.

Example:
Let $G = \left\{ \begin{bmatrix} e^{ir} & 0 & 0 \\ 0 & e^{is} & 0 \\ 0 & 0 & e^{it} \end{bmatrix} : r, s, t \in \mathbb{R} \right\} \subseteq GL(3, \mathbb{C})$

G is a mlg with Lie alg. $\mathfrak{g}$ given by

$$\mathfrak{g} = \left\{ \begin{bmatrix} ix & 0 & 0 \\ 0 & iy & 0 \\ 0 & 0 & iz \end{bmatrix} : x, y, z \in \mathbb{R} \right\} \simeq \mathbb{R}^3 \text{ as a vector sp.}$$

Then, $\forall$ pairs $X, Y \in \mathfrak{g}$, $[X, Y] = 0$

Also, $so(3) \simeq \mathbb{R}^3$ as a real vector sp.

$$so(3) = \operatorname{span}\left\{ \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}}_{B_1}, \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}}_{B_2}, \underbrace{\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{B_3} \right\}$$

<u>Check</u> : $[B_1, B_2] = B_3$, $[B_2, B_3] = B_1$, $[B_3, B_1] = B_2$

This gives another proof that the mlg G & $SO(3)$ are not isom.

<u>Example</u> : $$su(2) = \operatorname{span}\left\{ \underbrace{\frac{1}{2}\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}}_{A_1}, \underbrace{\frac{1}{2}\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}}_{A_2}, \underbrace{\frac{1}{2}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{A_3} \right\}$$

$so(3) = \operatorname{span}\{B_1, B_2, B_3\}$ as in prev. example

<u>Check</u> : $[A_1, A_2] = A_3$, $[A_2, A_3] = A_1$, $[A_3, A_1] = A_2$

The linear map $F : su(2) \to so(3)$ given by $F(A_i) = B_i$ for $i = 1, 2, 3$ is a Lie alg. isom.

Note: - The Lie alg. isomp. $F : \mathfrak{su}(2) \to \mathfrak{so}(3)$ is induced by a certain Lie gp. homomp. $\varphi : SU(2) \to SO(3)$

- Infact, the mlg $SU(2)$ & $SO(3)$ are not isomp.

Ppn: Let G_1, G_2 be mlg with Lie algs. $\mathfrak{g}_1, \mathfrak{g}_2$ resp.
Let $f : G_1 \to G_2$ be a Lie gp. homomp. Then f sends one-parameter gp. in G_1 to one-parameter gp. in G_2.
Moreover, $\forall\, v \in \mathfrak{g}_1$, $f(e^v) = e^{Df_I(v)}$

Pf: Let $v \in \mathfrak{g}_1$ & let $\alpha(t) = e^{tv}$ be a one-parameter gp. in G_1.
Since f is a gp. homomp., $\gamma(t) = f(\alpha(t)) = f(e^{tv})$ is a one-parameter gp. in G_2.

$$\gamma(t_1+t_2) = f(e^{(t_1+t_2)v}) = f(e^{t_1 v} \cdot e^{t_2 v}) = f(e^{t_1 v}) \cdot f(e^{t_2 v}) = \gamma(t_1)\cdot\gamma(t_2)$$

T'fore, $\gamma(t) = e^{tA}$ for some $A \in \mathfrak{gl}(n, \mathbb{C})$

$$\Rightarrow \quad A = \gamma'(0) = \frac{d}{dt}\Big|_{t=0} f(e^{tv}) = \frac{d}{dt}\Big|_{t=0} (f(\alpha(t)))$$

$$= Df_I(\alpha'(0)) = Df_I(V)$$

We conclude that $f(e^{tv}) = e^{t Df_I(v)}$, so putting $t = 1$ gives the result.

Cor : The Lie alg. of $\ker(f)$ equals $\ker(Df_I)$.

The Adjoint action

Def^n : Let G be a mlg with Lie alg. $\mathfrak{g}$.
We write $GL(\mathfrak{g})$ for the set of all invertible linear transformations from $\mathfrak{g}$ to itself.

Note : Since $\mathfrak{g}$ is a real vec. sp. with some dimension k, if we choose a basis $\mathcal{B} = \{v_1, v_2, \dots, v_k\}$ of $\mathfrak{g}$, then $GL(\mathfrak{g})$ can be identified with $GL(k, \mathbb{R})$ via

$$\begin{aligned} GL(\mathfrak{g}) &\to GL(k, \mathbb{R}) \\ T &\mapsto [T]_{\mathcal{B}} \end{aligned}$$

where for any linear transformation $T : \mathfrak{g} \to \mathfrak{g}$, $[T]_{\mathcal{B}}$ denotes the matrix w.r.t the basis $\mathcal{B}$.

Hence, $GL(\mathfrak{g})$ can be thought of as a matrix Lie gp.

Note: for each $g \in G$, $Ad_g : \mathfrak{g} \to \mathfrak{g}$ is given by $Ad_g(X) = gXg^{-1}$. Hence $Ad_g : \mathfrak{g} \to \mathfrak{g}$ is a vec. sp. isomp. i.e Ad_g is an element of $GL(\mathfrak{g})$.

Defn: We will denote by $Ad : G \to GL(\mathfrak{g})$ the map which sends g to Ad_g

Ppn: Let G be a mLg with Lie alg. $\mathfrak{g}$. Then

1. The map $Ad : G \to GL(\mathfrak{g})$ is a Lie gp. homomp.

2. For all $g \in G$ & all $X, Y \in \mathfrak{g}$, we have
$$[Ad_g(X), Ad_g(Y)] = Ad_g([X, Y])$$

That is, $Ad_g : \mathfrak{g} \to \mathfrak{g}$ is a Lie alg. homomp. (infact, isomp.)

Pf : 1. For all $g_1, g_2 \in G$ & all $X \in \mathfrak{g}$,

$$\mathrm{Ad}_{(g_1 g_2)}(X) = (g_1 g_2) X (g_1 g_2)^{-1} = g_1 (g_2 X g_2^{-1}) g_1^{-1}$$
$$= \mathrm{Ad}_{g_1}(\mathrm{Ad}_{g_2}(X))$$

So, $\mathrm{Ad}(g_1 g_2) = \mathrm{Ad}(g_1) \cdot \mathrm{Ad}(g_2)$, so Ad is a gp. homomp.

Also, $\mathrm{Ad} : G \to GL(\mathfrak{g})$ is cts., hence a Lie gp. homomp.

2. Pf 1. : follows from earlier ppn, since $C_g : G \to G$ is Lie gp. homomp. so its derivative $\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}$ is a Lie alg. homomp.

Pf 2 :
$$[\mathrm{Ad}_g(X), \mathrm{Ad}_g(Y)] = [gXg^{-1}, gYg^{-1}]$$
$$= gXg^{-1}gYg^{-1} - gYg^{-1}gXg^{-1}$$
$$= gXYg^{-1} - gYXg^{-1}$$
$$= g(XY - YX)g^{-1}$$
$$= \mathrm{Ad}_g([X, Y])$$

Note : The homomp. $\mathrm{Ad} : G \to GL(\mathfrak{g})$ is called the adjoint action of G on $\mathfrak{g}$.

$\underline{\text{Def}^n}$: We call a homomp. from a mlg G to $GL(n, \mathbb{R})$ an action (a linear action) of G on $\mathbb{R}^n$.

We write $G \curvearrowright \mathbb{R}^n$

eg: - $SO(n) \curvearrowright \mathbb{R}^n$ via left-multiplication corresponds to the inclusion $SO(n) \hookrightarrow GL(n, \mathbb{R})$

- Above discussion tells us that there is a natural action of $SO(n)$ on $\mathfrak{so}(n) \simeq \mathbb{R}^{n(n-1)/2}$

$\underline{\text{Def}^n}$: For any $X \in \mathfrak{g}$, we define the linear map $ad : \mathfrak{g} \to \mathfrak{g}$ by

$$\begin{aligned} ad_X : \mathfrak{g} &\to \mathfrak{g} \\ Y &\mapsto [X, Y] \end{aligned}$$

- If we choose a basis $\mathcal{B}$ for $\mathfrak{g}$, then the linear map ad_X is represented by a matrix A.

The linear transformation from $\mathfrak{g}$ to $\mathfrak{g}$ which is associated to the matrix e^A is denoted by e^{ad_X}

- Note that the linear transformation e^{ad_X} obtained in this way is independent of the choice of bases $\mathcal{B}_1$ & $\mathcal{B}_2$ resp, then $A_2 = CA_1C^{-1}$ for some inv'ble matrix C.

So, $e^{A_2} = e^{CA_1C^{-1}} = Ce^{A_1}C^{-1}$ so e^{A_2} & e^{A_1} represent the same linear transformation (wrt bases $\mathcal{B}_1, \mathcal{B}_2$ resp.)

Ppn: for all $X \in \mathfrak{g}$, $Ad_{e^X} = e^{ad_X}$

Pf: We have $Ad : G \to GL(\mathfrak{g})$

Since $Ad_{(e^X)} = Ad(e^X) = e^{D(Ad)_I(X)}$

by ppn

Lie gp. homomp. $G_1 \to G_2$, then $f(e^X) = e^{Df_I(X)}$

So, if we show that $D(Ad)_I(X) = ad_X$, then we are done.

Recall that, $\mathrm{ad}_X(Y) = [X,Y] = \left.\dfrac{d}{dt}\right|_{t=0} (\mathrm{Ad}_{e^{tX}}(Y))$

$$= \left.\frac{d}{dt}\right|_{t=0} (\mathrm{Ad}\circ\gamma(t))(Y) \quad , \quad \gamma(t) = e^{tX}$$

$$= \underbrace{D(\mathrm{Ad})_I(X)}(Y)$$

linear map $\mathfrak{g} \to \mathfrak{g}$
(element of $T_I(GL(\mathfrak{g}))$ i.e $\mathfrak{gl}(\mathfrak{g})$)

$$\Rightarrow \quad \mathrm{ad}_X = D(\mathrm{Ad})_I(X)$$

<u>Note</u>: $(e^{\mathrm{ad}_X})(Y) = \left(I + \mathrm{ad}_X + \frac{1}{2!}(\mathrm{ad}_X)^2 + \dots\right)(Y)$

$$= Y + [X,Y] + \frac{1}{2!}[X,[X,Y]] + \dots$$

So, we can interpret the ppn by saying that if $g = e^X$, then $\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}$ can be calculated completely in terms of repeated Lie brackets with X.

Def^n: Let G be a mlg with Lie alg $\mathfrak{g}$

A subspace $\mathfrak{h} \subset \mathfrak{g}$ is called a subalgebra if it is closed under the Lie bracket operation, i.e., if $[A, B] \in \mathfrak{h}$ whenever $A, B \in \mathfrak{h}$

eg: If $H < G$, then $\mathfrak{h}$ is a subalg. of $\mathfrak{g}$.

Def^n: A subalg. $\mathfrak{h} \subseteq \mathfrak{g}$ is called an ideal if $[A, B] \in \mathfrak{h}$ for all $A \in \mathfrak{h}$, $B \in \mathfrak{g}$

Pp^n: Let G be a path-conn. mlg, & let U be a nbd of I in G. Then, U generates G i.e every elem. in G is equal to a finite product $g_1 g_2 \dots g_k$ where for each i, either g_i or g_i^{-1} lies in U. (proved later*)

Thm: Let G be a path-conn. mlg, & let $H < G$ be a path-conn. subgp. Denote their Lie alg. as $\mathfrak{h} \subset \mathfrak{g}$. Then H is a normal subgp. of G iff $\mathfrak{h}$ is an ideal of $\mathfrak{g}$.

Pf: ($\Rightarrow$) Suppose H is a normal subgp. of G.

Let $A \in \mathfrak{h}$, $B \in \mathfrak{g}$. Let $a(t)$ be a path in H with $a(0) = I$ & $a'(0) = A$.

Let $b(t)$ be a path in G with $b(0) = I$ & $b'(0) = B$.

$$[A, B] = -[B, A] = -\frac{d}{dt}\Big|_{t=0} \left(Ad_{b(t)}(A)\right)$$

$$= -\frac{d}{dt}\Big|_{t=0} \left(b(t)\, A\, b(t)^{-1}\right)$$

$$= -\frac{d}{dt}\Big|_{t=0} \left(\frac{d}{ds}\Big|_{s=0} b(t)\, a(t)\, b(t)^{-1}\right)$$

Since H is normal in G, $b(t)\,a(t)\,b(t)^{-1} \in H$.

$$\Rightarrow \quad k(t) = \frac{d}{ds}\Big|_{s=0} \left(b(t)\,a(t)\,b(t)^{-1}\right) \in \mathfrak{h}$$

$$\Rightarrow \quad [A, B] = -\frac{d}{dt}\Big|_{t=0} k(t) \in \mathfrak{h}$$

$\Rightarrow [A, B] \in \mathfrak{h}$

Hence, $\mathfrak{h}$ is an ideal of $\mathfrak{g}$.

($\Leftarrow$) Suppose $\mathfrak{h}$ is an ideal of $\mathfrak{g}$. For every $B \in \mathfrak{g}$ & $A \in \mathfrak{h}$, we have $Ad_{e^B}(A) = e^{ad_B}(A)$

$$
\begin{aligned}
Ad_{e^B}(A) &= e^{ad_B}(A) \\
&= A + [B, A] + \frac{1}{2!}[B,[B,A]] + \frac{1}{3!}[B,[B,[B,A]]] + \dots
\end{aligned}
$$

where each term in the series in an elem. of $\mathfrak{h}$.

Hence, $Ad_{e^B}(A)$ is a limit of elems in $\mathfrak{h}$, hence $Ad_{e^B}(A) \in \mathfrak{h}$

Now, let $a \in H$, $b \in G$. Let U be a nbd of I in G s.t

$\exp : V \to U$ is a diffeomp. with $\exp : V \cap \mathfrak{h} \to U \cap H$ diffeomp.

$0 \in V$
$(\in \mathfrak{g})$

1) First, assume $a, b \in U$, s.t $a = e^A$ for some $A \in \mathfrak{h}$.

& $b = e^B$ for some $B \in \mathfrak{g}$.

Then, $bab^{-1} = be^A b^{-1} = e^{bAb^{-1}} = e^{Ad_b(A)} = e^{Ad_{e^B}(A)} \in H$

2) For general $a \in H$, $b \in G$ (<u>not</u> necessarily in U), by the pp^n,

$b = b_1 b_2 \cdots b_k$, $a = a_1 a_2 \cdots a_n$ where for each i, either b_i or b_i^{-1} lies in $U \cap G$ & for each j, either a_j or a_j^{-1} lies in $U \cap H$.

Then $bab^{-1} = (b_1 b_2 \cdots b_k)(a_1 a_2 \cdots a_n)(b_1 b_2 \cdots b_k)^{-1}$

$$= (b_1 b_2 \cdots b_{k-1})\,(\underbrace{b_k a_1 b_k^{-1}}_{\in H}\ \underbrace{b_k a_2 b_k^{-1}}_{\in H} \cdots \underbrace{b_k a_n b_k^{-1}}_{\in H})\,(b_{k-1}^{-1} \cdots b_2^{-1} b_1^{-1})$$

(by case 1)

$$= (b_1 \cdots b_{k-1})\ \underbrace{\tilde{a}}_{\in H}\ (b_{k-1}^{-1} \cdots b_1^{-1})$$

Keep repeating in this way to conclude that $bab^{-1} \in H$.

Some examples of Adjoint action

1. $G = SO(3)$

$$\mathfrak{g} = so(3) = \text{span}\left\{ \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}}_{B_1}, \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}}_{B_2}, \underbrace{\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{B_3} \right\}$$

Recall : $[B_1, B_2] = B_3$, $[B_2, B_3] = B_1$, $[B_3, B_1] = B_2$

We have a vector sp. isomp. $so(3) \cong \mathbb{R}^3$ via the map $B_i \mapsto e_i$

Consider $g = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \in SO(3)$

Then $Ad_g : so(3) \to so(3)$ satisfies $Ad_g(B_1) = g B_1 g^{-1}$

$$= \begin{pmatrix} 0 & 0 & -\sin\theta \\ 0 & 0 & -\cos\theta \\ \sin\theta & \cos\theta & 0 \end{pmatrix}$$

$$= \cos\theta \, B_1 - \sin\theta \, B_2$$

Sim., $Ad_g(B_2) = \sin\theta \, B_1 + \cos\theta \, B_2$

$Ad_g(B_3) = B_3$

So, $$[Ad_g]_{\mathcal{B}} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = g$$

We will show that, infact, with this choice of basis $\mathcal{B}$,

$[Ad_g]_{\mathcal{B}} = g \quad \forall \, g \in SO(3)$

For convenience of not^n, let us write $\tilde{Ad} : SO(3) \to GL(3, \mathbb{R})$.

Then for the map which sends $g \in SO(3)$ to $[Ad_g]_{\mathcal{B}}$

Ppn: Wrt. the basis $\mathcal{B}$, $\widetilde{Ad} : SO(3) \to GL(3,\mathbb{R})$ is the inclusion map $A \longmapsto A$

Pf:

Claim: $D(\widetilde{Ad})_I : so(3) \to gl(3,\mathbb{R})$ is given by $D(\widetilde{Ad})_I(X) = X$

Since $D(\widetilde{Ad})$ is a linear map, it is enough to prove the claim for a basis. Note that $ad_{B_1}(B_1) = [B_1, B_1] = 0$,

$$ad_{B_1}(B_2) = [B_1, B_2] = B_3,$$

$$ad_{B_1}(B_3) = [B_1, B_3] = -B_2$$

Hence, $[ad_{B_1}]_{\mathcal{B}} = B_1$.

Since $D(\widetilde{Ad})_I(X) = [ad_X]_{\mathcal{B}}$, we conclude $D(\widetilde{Ad})_I(B_1) = B_1$

Sim., $[ad_{B_2}]_{\mathcal{B}} = B_2$ & $[ad_{B_3}]_{\mathcal{B}} = B_3$, so

$D(\widetilde{Ad})_I(B_2) = B_2$ & $D(\widetilde{Ad})_I(B_3) = B_3$

We conclude that $D(\widetilde{Ad})_I(X) = X \quad \forall\ X \in so(3)$.

Claim: Wrt the basis $\mathcal{B}$ above, $\widetilde{Ad} : SO(3) \to GL(3;\mathbb{R})$ is the inclusion map. $A \mapsto A$

Let $B_r \subseteq so(3)$ & $U \subseteq SO(3)$ (open) be s.t $I \in U$ & s.t $\exp : B_r \to U$ is a diffeomp.

If $g \in U$, then $g = e^X$ for some $X \in so(3)$,

$$\Rightarrow \widetilde{Ad}(g) = [Ad_{e^X}]_{\mathcal{B}} = e^{D(\widetilde{Ad})_I(X)} = e^X = g \quad \text{(by prev. claim)}$$

So, $\widetilde{Ad} : U \to GL(3,\mathbb{R})$ is the map $g \mapsto g$

Let $S = \{ g \in SO(3) : \widetilde{Ad}(g) = g \} \subseteq SO(3)$

Since $SO(3)$ is conn., if we can show the set S is both open & closed, we will be done.

1) S is closed follows from the continuity of Ad (take seq. $\{g_n\}$ in S with $g_n \to g$)

2) S is open:

Let $g \in S$, so $\widetilde{Ad}(g) = g$

$L_g(U) = \{ g \cdot u : u \in U \}$ is an open set in $SO(3)$ containing g.

Then for any $v \in L_g(\mathcal{U})$, $v = g \cdot u$ for some $u \in \mathcal{U}$, so

$$\begin{aligned}\widetilde{Ad}(v) = \widetilde{Ad}(g \cdot u) = [Ad(g \cdot u)]_{\mathcal{B}} &= [Ad(g) \cdot Ad(u)]_{\mathcal{B}} \\ &= [Ad(g)]_{\mathcal{B}} [Ad(u)]_{\mathcal{B}} \\ &= \widetilde{Ad}(g) \cdot \widetilde{Ad}(u) \\ &= g \cdot u = v\end{aligned}$$

Hence, $L_g(\mathcal{U}) \subseteq S$, so S is open

This completes the proof.

***Pp^n**: Let G be a path-conn. mlg, & let $\mathcal{U}$ be a nbd of I in G.
Then, $\mathcal{U}$ generates G i.e every elem. in G is equal to a finite product $g_1 g_2 \dots g_k$ where for each i, either g_i or g_i^{-1} lies in $\mathcal{U}$.

Pf: Let $H = \langle \mathcal{U} \rangle$ i.e the subgp. of G generated by $\mathcal{U}$.
Since G is conn., if we show that H is both open & closed, then we're done.

Open: Let $h \in H$. Then $L_h(\mathcal{U}) = \{h \cdot u : u \in \mathcal{U}\}$ is open in G.
(image of open set $\mathcal{U}$ under the self-homeomp L_h)

Also, $h \in L_h(U)$ & $L_h(U) \subseteq H$.

So, $L_h(U)$ is a nbd of h in H, so H is open.

<u>Closed</u> : Let $\{h_n\}$ seq. in H with $h_n \to g$ in G.

$\Rightarrow h_n^{-1} \to g^{-1}$ (since $i : G \to G$ is cts.)
$x \mapsto x^{-1}$

$\Rightarrow g \cdot h_n^{-1} \to I$ (L_g is a cts. map)

$\Rightarrow \forall$ n sufficiently large, $g \cdot h_n^{-1} \in U$

$\Rightarrow g = \underbrace{g \cdot h_n^{-1}}_{\in U} \cdot \underbrace{h_n}_{\in H = \langle U \rangle}$, so $g \in H = \langle U \rangle$

Hence, H is closed.

(Examples of Adjoint action continued...)

2) <u>Adjoint action of $G = SU(2)$ on $\mathfrak{g} = su(2)$</u>

$$su(2) = \text{span}\Big\{ \underbrace{\tfrac{1}{2}\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}}_{A_1}, \underbrace{\tfrac{1}{2}\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}}_{A_2}, \underbrace{\tfrac{1}{2}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{A_3} \Big\}$$

basis $\mathcal{B}_1$

We have an inner product on $su(2)$ by $\langle A, B \rangle = 2\,tr(AB^*)$,
then the above basis $\mathcal{B}_1$ is an o.n.b for $su(2) \simeq \mathbb{R}^3$

For $g \in SU(2)$ & $A, B \in su(2)$, we have

$$\begin{aligned} \langle Ad_g(A), Ad_g(B) \rangle &= 2\,tr(gAg^{-1}gB^*g^{-1}) \\ &= 2\,tr(gAB^*g^{-1}) \\ &= 2\,tr(AB^*) = \langle A, B \rangle \end{aligned}$$

i.e the linear isomp. $Ad_g : su(2) \to su(2)$ preserves the inner product.

$\Rightarrow [Ad_g]_{\mathcal{B}_1} \in O(3)$

Let us denote by $\widetilde{Ad} : SU(2) \to GL(3,\mathbb{R})$, the map which sends $g \in SU(2)$ to $[Ad g]_{B_1}$, so what we observe is that $\widetilde{Ad} : SU(2) \to O(3)$

Infact, $SU(2)$ (homeomp. to S^3) is conn. & $\widetilde{Ad}$ is cts.,
so image$(\widetilde{Ad})$ lies in $SO(3)$ (i.e the path component of $O(3)$ which contains I).

So, all in all, we have a gp. homomp. $\widetilde{Ad} : SU(2) \to SO(3)$

Ppn: For $\widetilde{Ad} : SU(2) \to SO(3)$, $\ker(\widetilde{Ad}) = \{I, -I\}$.

Pf: $\{I, -I\} \subseteq \ker(\widetilde{Ad})$ is clear.
For the inverse inclusion, recall $SU(2) = \left\{ \begin{bmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{bmatrix} : |z_1|^2 + |z_2|^2 = 1 \right\}$

Let $g = \begin{bmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{bmatrix} \in \ker(\widetilde{Ad})$

Then $g \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} g^{-1} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$ implies $|z_1|^2 - |z_2|^2 = 1$ & $z_1 z_2 = 0$
from which we get $z_2 = 0$, $|z_1|^2 = 1$.

Then, from $g \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} g^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, we get that $z_1 = \pm 1$,
so $g = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

$\underline{\text{Ppn}}$: S^3 is path conn.

$\underline{\text{Pf}}$: We will show, for each $p \in S^3$, there is a path

$\alpha_p : [0,1] \to S^3$ s.t $\alpha_p(0) = p_0 = (1,0,0,0)$ & $\alpha_p(1) = p$

If $p = (-1,0,0,0)$, then let $\alpha_p(t) = (\cos \pi t, \sin \pi t, 0, 0)$

and if $p \neq (-1,0,0,0)$, then let $\alpha_p(t) = \dfrac{(1-t)p_0 + tp}{\|(1-t)p_0 + tp\|}$

Ppn: $D(\widetilde{Ad})_I : su(2) \to so(3)$ sends the basis $\mathcal{B}_1 = \{A_1, A_2, A_3\}$ of $su(2)$ to the basis $\mathcal{B}_2 = \{B_1, B_2, B_3\}$ of $so(3)$.

Pf: We have $D(\widetilde{Ad})_I(X) = [ad_X]_{\mathcal{B}_1}$

Note $ad_{A_1} : su(2) \to su(2)$ is the map given
$$ad_{A_1}(A_1) = [A_1, A_1] = 0$$
$$ad_{A_1}(A_2) = [A_1, A_2] = A_3$$
$$ad_{A_1}(A_3) = [A_1, A_3] = -A_2$$

So, $[ad_{A_1}]_{\mathcal{B}_1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = B_1 \in so(3)$

Sim., we can show that $[ad_{A_2}]_{\mathcal{B}_1} = B_2$, $[ad_{A_3}]_{\mathcal{B}_1} = B_3$

Hence, $D(\widetilde{Ad})_I(A_i) = B_i \quad \forall\ i = 1, 2, 3.$

Cor: $\exists$ a nbd U_I of I in $SU(2)$ s.t $\tilde{Ad}\big|_{U_I} : U_I \to SO(3)$ is a diffeomp. onto its img.

Pf: Since $D(\tilde{Ad})_I$ is inv'ble (sends basis to basis)
The result follows by the Inverse fx^n thm for smooth maps b/w manifolds.

Def^n: Let M, N be manifolds & let $f: M \to N$ be a smooth map. We will say that f is a local diffeomp. if $\forall\, p \in M$, $\exists$ nbd U of p in M & a nbd V of $f(p)$ in N s.t $f: U \to V$ is a diffeomp.

eg: $f: S^1 \to S^1$ gives by $z \mapsto z^2$
(local diffeomp. but not diffeomp.)

$\underline{\text{Ppn}}$: $\widetilde{Ad} : SU(2) \to SO(3)$ is a local diffeomp.

$\underline{\text{Pf}}$: Recall,

HW 5.3: Let $f : G_1 \to G_2$ be a Lie gp. homomp.

Suppose $Df_I : \mathfrak{g}_1 \to \mathfrak{g}_2$ is bij. Then $Df_g : T_gG_1 \to T_gG_2$ is bij.

Since we have proved $D(\widetilde{Ad})_I$ is bij, so HW 5.3 implies that $D(\widetilde{Ad})_g$ is also bij.

Then, using the Inverse Fxn thm as above, we conclude that $\exists$ nbd U_g of g in $SU(2)$ s.t $\widetilde{Ad}_{U_g} : U_g \to SO(3)$ is a diffeomp. onto its image.

$\underline{\text{Ppn}}$: $Ad : SU(2) \to SO(3)$ is surjective

$\underline{\text{Pf}}$: Since $SU(2)$ is compact, its image under $\widetilde{Ad}$ is compact, and hence closed.

We will show that image is also open, which will prove the statement, since $SO(3)$ is conn.

Let $a \in \text{Image}(\widetilde{Ad}) \subseteq SO(3)$ with $g \in SU(2)$ s.t $\widetilde{Ad}(g) = a$.
By prev. pp^n, $\widetilde{Ad}$ is a local diffeomp, which gives us open nbd U of g in $SU(2)$, open nbd V in $SO(3)$ s.t $\widetilde{Ad}(U) = V$

Hence, $a \in V \subseteq \text{Image}(\widetilde{Ad})$, so $\text{Image}(\widetilde{Ad})$ is open.

Summary

- As a group, $SO(3)$ is isomp. to $SU(2)/\{I, -I\}$

- $\widetilde{Ad}: SU(2) \to SO(3)$ is a surj, 2-to-1 local diffeomp.
(a double cover)

- $SO(3)$ can be viewed a quotient sp. obtained from the 3-sphere S^3 via identifying pairs of antipodal pts. on S^3.

- Each coset of $\text{Ker}(\widetilde{Ad}) = \{I, -I\}$ in $SU(2)$ is of the form $\{g, -g\}$ for $g \in SU(2)$ & via the bij.

$$SU(2) \longrightarrow S^3 = \{ V \in \mathbb{R}^3 : \|V\| = 1 \}$$

$$g = \begin{bmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{bmatrix} \longmapsto (x_1, y_1, x_2, y_2) = V$$

The set $\{g, -g\}$ gets sent to the set $\{v, -v\} \subseteq \mathbb{R}^4$

Defⁿ: The set of all lines through the origin in $\mathbb{R}^{n+1}$ is called the n-dimensional real projective sp. & is denoted as $\mathbb{RP}^n$

- $\mathbb{RP}^n$ is a topological sp. it has the quotient topo. induced by viewing it as a quotient sp. of $\mathbb{R}^{n+1} \setminus \{0\}$ via the equivalence relⁿ for any $v \in \mathbb{R}^{n+1} \setminus \{0\}$, $v \sim \lambda v$ for all $\lambda \in \mathbb{R}, \lambda \neq 0$

- Every line through the origin in $\mathbb{R}^{n+1}$ intersects the sphere S^n in a pair of antipodal pts, and $\mathbb{RP}^n$ can be equivalently defined as the quotient sp. obtained from S^n via the equivalence relⁿ $v \sim -v \quad \forall \ v \in S^n$.

- We have shown that $SO(3)$ & $\mathbb{RP}^3$ are homeomp.

Rem: This does not yet prove that $SU(2)$ & $SO(3)$ are not isomp.

In Algebraic Topology, one associates to topological sp. an invariant called the fundamental grp., which can be used to distinguish S^3 & $\mathbb{RP}^3$.

- So far, we've been going from Lie gp. to Lie alg.

- There is a certain sense in which you can go backwards i.e 'recover' the product operation on this gp. from the Lie bracket

- The main tool for this will be Baker-Campbell-Hausdorff theorem, which (for $X, Y \in \mathfrak{gl}(n, \mathbb{C})$ with sufficiently small norm) expresses $\log(e^X e^Y)$ in terms of X, Y & repeated Lie brackets of X & Y.

- We will use this (under some assumptions), to go from a Lie alg. homomp $f : \mathfrak{g} \to \mathfrak{h}$ to a Lie gp. homomp $F : G \to H$

- We'll see all this first in the special case of the Heisenberg gp.

Thm: Let $X, Y \in M_n(\mathbb{C})$ be matrices satisfying

$$[X,[X,Y]] = [Y,[X,Y]] = 0 \qquad -(*)$$

Then $e^X \cdot e^Y = e^{X+Y+\frac{1}{2}[X,Y]}$

Pf: We will show that $e^{tX} \cdot e^{tY} = e^{(tX+tY+\frac{t^2}{2}[X,Y])}$ which gives the required result by putting $t=1$.

By assumption $(*)$, $\frac{t^2}{2}[X,Y]$ commutes with $t(X+Y)$, so, the expression is equivalent to

$$e^{tX} \cdot e^{tY} \, e^{-\frac{t^2}{2}[X,Y]} = e^{t(X+Y)}$$

Def. $A(t) = e^{tX} e^{tY} e^{-\frac{t^2}{2}[X,Y]}$

$B(t) = e^{t(X+Y)}$

Note that, $A(0) = I = B(0)$. We will show that $A(t)$ & $B(t)$ satisfy the same differential eq^n.

First, note that $\frac{dB}{dt} = B(t)(X+Y)$

Next, product rule implies that

$$\frac{dA}{dt} = e^{tX} X e^{tY} e^{-\frac{t^2}{2}[X,Y]} + e^{tX} e^{tY} Y e^{-\frac{t^2}{2}[X,Y]} + e^{tX} e^{tY} e^{-\frac{t^2}{2}[X,Y]} (-t[X,Y])$$

Since Y commutes with $[X,Y]$, it also commutes with $e^{-\frac{t^2}{2}[X,Y]}$.
Hence, the second term in $\frac{dA}{dt}$ equals $e^{tX} \cdot e^{tY} e^{-\frac{t^2}{2}[X,Y]}$

For the first term of dA/dt, write

$$X e^{tY} = e^{tY} e^{-tY} X e^{tY} = e^{tY} Ad_{e^{-tY}}(X) = e^{tY} e^{ad_{(-tY)}}(X)$$

$$= e^{tY} e^{-t\,ad_Y}(X)$$

Now, $e^{-t\,ad_Y}(X) = X - t[Y,X] + \frac{t^2}{2}[Y,[Y,X]] + \ldots$

But $[Y,[Y,X]] = -[Y,[X,Y]] = 0$ (assumption in thm statement)

So, all higher order terms in this series vanish, & we have

$$e^{-t\,ad_Y}(X) = X - t[Y,X]$$
$$= X + t[X,Y]$$

Hence, first term in dA/dt is

$$e^{tX} X e^{tY} e^{-\frac{t^2}{2}[X,Y]} = e^{tX} e^{tY} (X + t[X,Y])\, e^{-\frac{t^2}{2}[X,Y]}$$
$$= e^{tX} e^{tY} e^{-\frac{t^2}{2}[X,Y]} (X + t[X,Y])$$

Hence, $$\frac{dA}{dt} = e^{tX}e^{tY}e^{-\frac{t^2}{2}[X,Y]}(X + t[X,Y]) + e^{tX}e^{tY}e^{-\frac{t^2}{2}[X,Y]}Y + e^{tX}e^{tY}e^{-\frac{t^2}{2}[X,Y]}(-t[X,Y])$$

$$= A(t)(X+Y)$$

Since $A(t)$ & $B(t)$ satisfy the same ODE with $A(0) = I = B(0)$, uniqueness theorem of ODEs implies that $A(t) = B(t)$ $\forall t$.

<u>Ppn</u>: Let H denote the Heisenberg gp. & $\mathfrak{h}$ its Lie alg. Then $\exp : \mathfrak{h} \to H$ is one-one & onto.

<u>Pf</u>: Note, $$\mathfrak{h} = \left\{ \begin{bmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{bmatrix} : x, y, z \in \mathbb{R} \right\}$$

If $X = \begin{bmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{bmatrix}$, then $X^2 = \begin{bmatrix} 0 & 0 & xz \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $X^n = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\forall n \geq 3$

Hence, $$e^X = \begin{pmatrix} 1 & x & y + \frac{xz}{2} \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}$$

<u>One-one</u>: $$e^{X_1} = e^{X_2} \Rightarrow \begin{pmatrix} 1 & x_1 & y_1 + \frac{x_1 z_1}{2} \\ 0 & 1 & z_1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x_2 & y_2 + \frac{x_2 z_2}{2} \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix}$$

So, $x_1 = x_2$, $z_1 = z_2$, $y_1 + \frac{x_1 z_1}{2} = y_2 + \frac{x_2 z_2}{2} \Rightarrow y_1 = y_2$

$\Rightarrow X_1 = X_2$

So, $\exp : \mathfrak{h} \to H$ is one-one.

<u>Onto</u>: Let $A = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \in H$.

Define $x = a$, $z = c$, $y = b - \frac{ac}{2}$, then $X = \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix}$

satisfies $e^X = A$

<u>Thm</u>: Let H denote the Heisenberg gp. & $\mathfrak{h}$ its Lie alg.
Let G be a mlg with Lie alg. $\mathfrak{g}$. Let $f : \mathfrak{h} \to \mathfrak{g}$ be a Lie alg. homomp. Then, there exists a unique Lie gp. homomp. $F : H \to G$ s.t $F(e^X) = e^{f(X)}$ $\forall X \in \mathfrak{h}$

<u>Pf</u>: Since $\exp : \mathfrak{h} \to H$ is bij., let $\log : H \to \mathfrak{h}$ denote its inverse. Def. $F : H \to G$ by $F(A) = e^{f(\log A)}$

We will show that F is a Lie gp. homomp.

1. Continuity of F is easy ($\log$, f, $\exp$ are cts.)

2. If $X, Y \in \mathfrak{h}$, then $[X, Y]$ is of the form $\begin{bmatrix} 0 & 0 & z \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ &
such a matrix commutes with each elem in $\mathfrak{h}$, in particular with X & Y.

Thus, in this case, $[X, [X, Y]] = 0 = [Y, [X, Y]]$
Since f is a Lie alg. homomp., $f(X)$ & $f(Y)$ will also commute with $[f(X), f(Y)]$

$$[f(X),[f(X),f(Y)]] = [f(X), f([X,Y])]$$
$$= f([X,[X,Y]])$$
$$= f(0) = 0$$

Now for $A, B \in H$, $\exists X, Y \in \mathfrak{h}$ s.t $A = e^X$, $B = e^Y$.

By the previous theorem,

$$F(AB) = F(e^X e^Y) = F(e^{X+Y+\frac{1}{2}[X,Y]})$$
$$= e^{f(\log(e^{X+Y+\frac{1}{2}[X,Y]}))}$$
$$= e^{f(X+Y+\frac{1}{2}[X,Y])}$$
$$= e^{f(X)+f(Y)+\frac{1}{2}[f(X),f(Y)]}$$
$$= e^{f(X)} e^{f(Y)}$$
$$= e^{f(\log A)} e^{f(\log B)}$$
$$= F(A) F(B)$$

Hence, F is a group homomp.

Some remarks on the result proved last time

- For $H=$ Heisenberg gp. & G any mlg, we showed if $f: \mathfrak{h} \to \mathfrak{g}$ is a Lie alg homomp., then $\exists F: H \to G$ a Lie gp. homomp. s.t

$$e^{DF_I(X)} = F(e^X) = e^{f(X)} \quad \forall X \in \mathfrak{h}$$

- In the proof, we used that if $[X,[Y,X]] = [Y,[X,Y]] = 0$, then $e^X e^Y = e^{X+Y+\frac{1}{2}[X,Y]}$.

If X & Y are small enough, then this can be restated as

$$\log(e^X e^Y) = X + Y + \frac{1}{2}[X,Y]$$

- $\text{RHS} = X + Y + \frac{1}{2}XY - \frac{1}{2}YX$ is a polynomial $P(X,Y)$ consisting of linear combinations of X & Y & brackets of X & Y.

- Applying a Lie alg homomp. to such a polynomial we get $f(P(X,Y)) = P(f(X), f(Y))$

- Using this, we concluded,

$$e^{f(X)} \cdot e^{f(Y)} = e^{P(f(X), f(Y))}$$

i.e $\log(e^{f(X)} \cdot e^{f(Y)}) = P(f(X), f(Y))$

- To do this more generally, we want an expression for $\log(e^X \cdot e^Y)$ consisting of linear combinations of X, Y & repeated brackets.

Thm: For $A \in M_n(\mathbb{C})$ with $\|A - I\| < 1$, $\log A$ is given by a conv. power series $\log A = \sum_{m=1}^{\infty} (-1)^{m+1} \frac{(A-I)^m}{m}$

Pf: Skipped. See [Hall]

Baker – Campbell – Hausdorff thm

Goal: If $e^X \cdot e^Y = e^Z$, write Z as a fn of X, Y (implicitly assuming such a Z exists, true if X & Y of small enough norm)

We have,
$$e^X = I + X + \frac{X^2}{2!} + \frac{X^3}{3!} + \frac{X^4}{4!} + \dots$$
$$e^Y = I + Y + \frac{Y^2}{2!} + \frac{Y^3}{3!} + \frac{Y^4}{4!} + \dots$$

So,
$$e^X \cdot e^Y = I + X + Y + XY + \frac{X^2}{2!} + \frac{Y^2}{2!} + \dots$$

Then,
$$Z = \log(e^X \cdot e^Y) = \left(X + Y + XY + \frac{X^2}{2!} + \frac{Y^2}{2!} + \dots \right)$$
$$- \frac{1}{2} \left(X + Y + XY + \frac{X^2}{2!} + \frac{Y^2}{2!} + \dots \right)^2$$
$$+ \frac{1}{3} \left(X + Y + XY + \frac{X^2}{2!} + \frac{Y^2}{2!} + \dots \right)^3$$
$$= X + Y + \frac{1}{2} XY - \frac{1}{2} YX + \text{higher order terms}$$

$\uparrow$

rearrangement justified by absolute conv. of both series within radius of conv.

$$= X + Y + \frac{1}{2}[X, Y] + \text{higher order terms}$$

Write $F_n(X,Y)$ for the sum of all terms of degree n in the above series i.e $Z = \log(e^X . e^Y) = \sum_{n\geq 1} F_n(X,Y)$

By the above discussion, $F_1(X,Y) = X + Y$

$$F_2(X,Y) = \frac{1}{2}[X,Y]$$

See [Stillwell], $F_3(X,Y) = \frac{1}{12}\left(X^2Y + XY^2 + YX^2 + Y^2X - 2XYX - 2YXY\right)$

is a linear combination of $[X,[X,Y]]$ & $[Y,[X,Y]]$.

Defn: We will say that a polynomial $p(A,B,C,\dots)$ is a Lie polynomial if it is a linear combination of $A, B, C \dots$, and repeated Lie bracket terms in $A, B, C \dots$

eg: $F_1(A,B)$ & $F_2(A,B)$ are Lie polynomials.

Thm: (Baker - Campbell - Hausdorff)

If $Z = \log(e^A \cdot e^B) = \sum_{n\geq 1} F_n(A,B)$, where $F_n(A,B)$ is the sum of all terms of degree n, then for each $n \geq 1$, the polynomial $F_n(A,B)$ is a Lie polynomial.

Pf: We will prove by ind^n.

First since $(e^A e^B) e^C = e^A (e^B e^C)$, if $e^A e^B e^C = e^W$, then

$$W = \sum_{i=1}^{\infty} F_i\left(\sum_{j=1}^{\infty} F_j(A,B), C\right) = \sum_{i=1}^{\infty} F_i\left(A, \sum_{j=1}^{\infty} F_j(B,C)\right) \quad -(1)$$

Ind^n Hyp: F_m is a Lie poly. for $m < n$

Claim: F_n is a Lie poly.

Ind^n hyp. $\Rightarrow$ all homogeneous terms of degree $< n$ in both expressions for W are Lie poly.

Also, ind^n hyp. $\Rightarrow$ all homogeneous terms of degree n coming from $i > 1$ & $j > 1$ are Lie poly., except possibly

a) $F_n(A,B) + F_n(A+B, C)$ on the left (coming from $i=1, j=n$ & $i=n, j=1$)

b) $F_n(A, B+C) + F_n(B,C)$ on the right (coming from $i=n, j=1$ & $i=1, j=n$)

Equating terms of degree n on both sides of (1), we see that the difference of the terms in (a) & (b) must be a Lie poly.

This defines a convergence relⁿ ($\sim$) b/w poly. & we write it as:

$$F_n(A,B) + F_n(A+B,C) \sim F_n(A,B+C) + F_n(B,C) \quad -(2)$$

(we say $P_1(X,Y) \sim P_2(X,Y)$ if $P_1(X,Y) - P_2(X,Y)$ is Lie)

Strategy: By substituting suitable values of A, B, C in (2), we will ultimately derive $F_n(A,B) \sim 0$, which will prove that F_n is Lie.

Three general facts.

i) For $r, s \in \mathbb{R}$, we have $F_n(rA, sA) = 0 \quad (n>1)$

because rA, sA commute, so $e^{rA} \cdot e^{sA} = e^{rA+sA}$

so $Z = rA + sA = F_1(rA, sA)$ & so all other

$F_n(rA, sA) = 0 \quad (n>1)$

ii) In particular, (putting $r=1, s=0$, resp. $r=0, s=1$), we get $F_n(A,0) = 0 = F_n(0,A)$

iii) $F_n(\lambda A, \lambda B) = \lambda^n F_n(A, B)$ because F_n is homogeneous of degree n.

Before, we'll assume $n > 1$.

First, replace C by $-B$ in (2)

$$F_n(A, B) + F_n(A+B, -B) \sim F_n(A, 0) + F(B, -B)$$
$$\sim 0 \quad (\text{by (ii) \& (i)})$$

T'fore, $F_n(A, B) \sim -F_n(A+B, -B)$ $\quad -(3)$

Then replace A by $-B$ in (2).

$$F_n(-B, B) + F_n(0, C) \sim F_n(-B, B+C) + F_n(B, 0)$$

By (i), (ii), this implies

$$0 \sim F_n(-B, B+C) + F_n(B, C)$$

Then, replace B, C by A, B resp.

$$0 \sim F_n(-A, A+B) + F_n(A, B)$$

So, $F_n(A, B) \sim -F_n(-A, A+B)$ $\quad -(4)$

Next, we will relate $F_n(A,B)$ to $F_n(B,A)$:

$$\begin{aligned}
F_n(A,B) &\sim -F_n(-A, A+B) \quad \text{(by (4))} \\
&\sim -(-F_n(-A+A+B, -A-B)) \quad \text{(by (3))} \\
&\sim F_n(B, -A-B) \\
&\sim -F_n(-B, -A) \quad \text{(by (4))} \\
&\sim -(-1)^n F_n(B,A)
\end{aligned}$$

Thus, $F_n(A,B) \sim -(-1)^n F_n(B,A) \quad -(5)$

Next, replace C by $-B/2$ in (2)

$$\begin{aligned}
F_n(A,B) + F_n(A+B, B/2) &\sim F_n(A, B/2) + F_n(B, -B/2) \\
&\sim F_n(A, B/2) \quad \text{(by (i))}
\end{aligned}$$

Hence, $F_n(A,B) \sim F_n(A, B/2) - F_n(A+B, -B/2) \quad -(6)$

Next, replace A by $-B/2$ in (2)

$$F_n(-B/2, B) + F_n(B/2, C) \sim F_n(-B/2, B+C) + F_n(B,C)$$

$$\Rightarrow F_n(B/2, C) \sim F_n(-B/2, B+C) + F_n(B,C) \quad \text{(by (i))}$$

Then, replacing B, C by A, B resp.

$$F_n(A/2, B) \sim F_n(-A/2, A+B) + F_n(A, B)$$

$$\Rightarrow \quad F_n(A, B) \sim F_n(A/2, B) - F_n(-A/2, A+B) \quad - (7)$$

We can rewrite the two terms on RHS of (7) as follows:

$$\begin{aligned} F_n(A/2, B) &\sim F_n(A/2, B/2) - F_n(A/2+B, -B/2) && (\text{by } (6)) \\ &\sim F_n(A/2, B/2) + F_n(A/2+B/2, B/2) && (\text{by } (3)) \\ &\sim 2^{-n} F_n(A, B) + 2^{-n} F_n(A+B, B) && (\text{by } (iii)) \end{aligned}$$

$$\begin{aligned} F_n(-A/2, A+B) &\sim F_n(-A/2, A/2+B/2) - F_n(A/2+B, -A/2-B/2) && (\text{by } (6)) \\ &\sim -F_n(A/2, B/2) + F_n(B/2, A/2+B/2) && (\text{by } (4) \,\&\, (3)) \\ &\sim -2^{-n} F_n(A, B) + 2^{-n} F_n(B, A+B) && (\text{by } (iii)) \end{aligned}$$

So, (7) becomes

$$F_n(A, B) \sim 2^{1-n} F_n(A, B) + 2^{-n} F_n(A+B, B) + 2^{-n} F_n(B, A+B)$$

Rearranging & using (5), this becomes

$$(1-2^{1-n}) F_n(A, B) \sim 2^{-n}(1+(-1)^n) F_n(A+B, B) \quad - (8)$$

If n is odd $(n>1)$, then (8) implies $F_n(A,B) \sim 0$

If n is even, we replace A by $A-B$ in (8)

$$(1-2^{1-n}) F_n(A-B, B) \sim 2^{1-n} F_n(A,B) \quad -(9)$$

LHS of (9):

$$(1-2^{1-n}) F_n(A-B,B) \sim -(1-2^{1-n}) F_n(A,-B) \quad (\text{by } (3))$$

Substituting this back in (9), we get

$$-F_n(A,-B) \sim \frac{2^{1-n}}{1-2^{1-n}} F_n(A,B) \quad -(10)$$

Finally replace B by $-B$ in (10),

$$-F_n(A,B) \sim \frac{2^{1-n}}{1-2^{1-n}} F_n(A,-B)$$

$$\sim \left(\frac{2^{1-n}}{1-2^{1-n}}\right)^2 F_n(A,B)$$

This implies that $F_n(A,B) \sim 0$ as required.

Ppn: Let G, K be mlgs with Lie algebras $\mathfrak{g}, \mathfrak{k}$ resp.

Let $f_1, f_2 : \mathfrak{g} \to \mathfrak{k}$ be linear maps s.t $e^{f_1(X)} = e^{f_2(X)}$ $\forall X \in \mathfrak{g}$

Then $f_1(X) = f_2(X)$ $\forall X \in \mathfrak{g}$

Pf: Let $r > 0$ s.t $\exp : B_r \to V$ is a diffeomp. where $B_r = \{ Y \in \mathfrak{k} : \|Y\| < r \}$ & $V \subseteq K$ open.

Since $f_1, f_2 : \mathfrak{g} \to \mathfrak{k}$ linear, they are cts.

$\Rightarrow \exists B_{r_1} \subseteq \mathfrak{g}$ s.t $f_1(B_{r_1}) \subseteq B_r$, $f_2(B_{r_2}) \subseteq B_r$

This implies that $\forall X \in B_{r_1}$, $f_1(X) = f_2(X)$
(since by assumption, $e^{f_1(X)} = e^{f_2(X)}$ & exp is one-one on B_r)

Finally, for arbitrary $X \in \mathfrak{g}$ $(X \neq 0)$

$\tilde{X} = \frac{r_1 X}{2\|X\|} \in B_{r_1}$, so $f_1(\tilde{X}) = f_2(\tilde{X})$

i.e $\frac{r_1}{2\|X\|} f_1(X) = \frac{r_1}{2\|X\|} f_2(X) \Rightarrow f_1(X) = f_2(X)$

Using the above ppn & the fact that for any homomp. F of mlg,
we have $F(e^X) = e^{DF_I(X)}$, we can restate last week's result as
follows

Thm: for H = Heisenberg gp. & G any mlg, if $f : \mathfrak{h} \to \mathfrak{g}$ is
a Lie alg. homomp., $\exists F : H \to G$ Lie gp. homomp. s.t $DF_I = f$.

Rem:

- In some sense, this is saying that knowing the derivative of F at one pt. (DF_I), you can recover F.

- Proof used that for H = Heisenberg gp.

1) $\exp : \mathfrak{h} \to H$ is bijective, so $\log$ is defined uniquely on all of H, $\log : H \to \mathfrak{h}$

2) If $Z = \log(e^X e^Y)$, then $Z = X + Y + \frac{1}{2}[X, Y]$

If we want to replace the Heisenberg gp. by an arbitrary mlg in the above thm, we have a generalization of (2) (BCH thm)

(1) is <u>not</u> true for general mLg (see Midsem Q, HW6 for counterexamples)

However, there is a topological assumption which can help us overcome this difficulty.

<u>Def^n</u>: A mLg G is said to be <u>simply connected</u> if it is path-conn. & in add^n, every loop in G can be deformed continuously to a pt. in G

More precisely, if G is path-conn., we say that G is simply conn., if for every cts. path $\alpha : I \to G$ with $\alpha(0) = \alpha(1)$ ($= P_0$ say), $\exists$ a cts. fx^n $A : I \times I \to G$ with the following ppts: ($I = [0,1]$)

1. $A(s,0) = A(s,1) = p_0 \quad \forall\, s \in I$
2. $A(0,t) = \alpha(t) \quad \forall\, t \in I$
3. $A(1,t) = p_0 \quad \forall\, t \in I$

eg: (without proof)

- $SO(2)$ is <u>not</u> simply conn.
- $SU(2)$ is simply conn.
- H = Heisenberg gp. is simple conn.

- $SO(3)$ is <u>not</u> simply conn.

- $G = \left\{ \begin{bmatrix} e^{it} & 0 \\ 0 & e^{is} \end{bmatrix} : t, s \in \mathbb{R} \right\}$ is <u>not</u> simply conn.

- In Algebraic topology, the failure of a topo. sp. to be simply conn. is measured in terms of an invariant called the fundamental gp. of the space $\pi_1(X)$

<u>Thm</u>: (Existence of a Lie Gp. homomp. given a Lie alg. homomp.)
Let G & H be mlgs with Lie algs. $\mathfrak{g}$ & $\mathfrak{h}$ resp.
Let $f : \mathfrak{g} \to \mathfrak{h}$ be a Lie alg. homomp. If G is simply-conn.,
$\exists$ a unique Lie gp. homomp. $F : G \to H$ s.t $F(e^X) = e^{f(X)}$ $\forall X \in \mathfrak{g}$
or equivalently, s.t $DF_I = f$

We will see some ideas of the proof (later).
First, an important corollary of this thm.

Cor: Let G, H be simply-conn. mlgs with Lie algs $\mathfrak{g}, \mathfrak{h}$ resp. If $\mathfrak{g}$ is isomp. to $\mathfrak{h}$ (as Lie alg.), then G is isomp. to H

Pf: Let $\varphi : \mathfrak{g} \to \mathfrak{h}$ be a Lie alg. isomp.
By the above thm, $\exists\ \Phi : G \to H$ Lie gp. homomp. with $D\Phi_I = \varphi$

Let $\psi : \mathfrak{h} \to \mathfrak{g}$ be the inverse of φ & let $\Psi : H \to G$ Lie gp. homomp. with $D\Psi_I = \psi$.

Then, $\Psi \circ \Phi : G \to G$ is a Lie gp. homomp. with
$D(\Psi \circ \Phi)_I = \psi \circ \varphi = Id : \mathfrak{g} \to \mathfrak{g}$

Claim: $\Psi \circ \Phi = Id : G \to G$

Pf: Let $B_r \subseteq \mathfrak{g}$, $V \subseteq G$ be s.t $\exp : B_r \to V$ is a diffeomp.
Since the map $\imath : G \to G$ given by $A \mapsto A^{-1}$ is a homeomp., we can choose $\tilde{V} \subset V$ open s.t $\imath(\tilde{V}) \subseteq \tilde{V}$.

Since G is path-conn., $\tilde{V}$ generates G. Thus, for any $g \in G$, $\exists\ X_1, X_2, \dots, X_k \in B_r \subset \mathfrak{g}$ s.t $g = e^{X_1} \cdots e^{X_k}$

Write $f = \Psi \circ \Phi$.

Then $f(g) = f(e^{X_1} \dots e^{X_k}) = f(e^{X_1}) \dots f(e^{X_k})$

$$= e^{Df_I(X_1)} \dots e^{Df_I(X_k)}$$

$$= e^{X_1} \dots e^{X_k} = g$$

This proves the claim.

Sim., $\Phi \circ \Psi = Id : H \to H$

This shows that Φ, Ψ are inverses of each other, so

G, H are isomp.

For proving the existence thm, we will first introduce a def^n

Def^n: If G, H are mlgs, a local homomp. of G to H is a pair (U, f) where U is a path-conn. nbd of I in G & $f : U \to H$ is a cts. map s.t $f(AB) = f(A)f(B)$ where A, B & AB all belong to U.

Ppⁿ: Let G & H be mlgs with Lie algebras $\mathfrak{g}, \mathfrak{h}$ resp.
Let $\varphi: \mathfrak{g} \to \mathfrak{h}$ be a Lie alg. homomp. Then $\exists$ a nbd $\mathcal{U}$ of I in G s.t the map $F: \mathcal{U} \to H$ given by $F(A) = e^{\varphi(\log A)}$ is a local homomp.

Pf: Choose $\mathcal{U}$ small enough s.t

1. $\log : \mathcal{U} \to \mathfrak{g}$ is uniquely defined
2. $\forall A, B \in \mathcal{U}$, the BCH thm applies to $X = \log(A)$ & $Y = \log(B)$, as well as to $\varphi(X)$ & $\varphi(Y)$

Then, if AB is in $\mathcal{U}$, we have (by the defⁿ of F)

$$F(AB) = F(e^X e^Y) = e^{\varphi(\log(e^X e^Y))}$$

By the BCH thm,
$$\varphi(\log(e^X e^Y)) = \varphi\Big(\sum_{n\geq 1} F_n(X, Y)\Big)$$
$$= \sum_{n\geq 1} F_n(\varphi(X), \varphi(Y))$$
$$= \log\big(e^{\varphi(X)} e^{\varphi(Y)}\big)$$

Hence, $F(AB) = e^{\log(e^{\varphi(X)} e^{\varphi(Y)})} = e^{\varphi(X)} e^{\varphi(Y)} = F(A)F(B)$
and so, $F: \mathcal{U} \to H$ defines a local homomp.

The next step is to show if G is simply conn., then a local homomp. of G into H can be extended uniquely to Lie gp. homomp. $\Phi : G \to H$

<u>Thm</u>: Let G, H be mLgs, with G simply conn.
If (U, F) is a local homomp. of G into H, $\exists$ a unique Lie gp. homomp. $\Phi : G \to H$ s.t Φ agrees with F on U.

This is the main technical step. Some ideas of the proof later.

First, proof of existence thm provided G simply conn.

<u>Proof</u>: Let F be the local homomp. given by the ppn, and (since G is simply conn.), let Φ be the Lie gp. homomp given by the thm above.

Then, for any $X \in \mathfrak{g}$, you can choose m large s.t the elem $e^{X/m}$ will be in U, and so

$$\Phi(e^{X/m}) = F(e^{X/m}) = e^{\varphi(X)/m}$$

Since Φ is a gp. homomp., we have

$$\Phi(e^X) = (\Phi(e^{X/m}))^m = (e^{\varphi(X)/m})^m = e^{\varphi(X)}$$

For uniqueness, suppose Φ_1 & Φ_2 are two Lie gp. homomps.
with $\Phi_i(e^X) = e^{\varphi(X)}$ $\forall X \in \mathfrak{g}$

Then, for any $A \in G$, we can write

$$A = e^{X_1} \cdots e^{X_N} \text{ with } X_i \in \mathfrak{g}$$

(this step uses that since G is path-conn., it is generated by any nbd of I)

So, $\Phi_1(A) = \Phi_2(A) = e^{\varphi(X_1)} \cdots e^{\varphi(X_N)}$

Hence, $\Phi_1 = \Phi_2$

Ideas in the proof

Step 1: Define Φ along a path.

Since G is simply-conn., it is path-conn., hence given any $A \in G$, $\exists$ path $A(t)$ in G with $A(0) = I$ & $A(1) = A$

We call a partition $0 = t_0 < t_1 < t_2 \dots < t_m = 1$ of $[0,1]$ a 'good partition' if $\forall s,t$ belonging to the same subinterval, $A(t)A(s)^{-1} \in \mathcal{U}$.

[Good partitions exist]

If a partition is good, since $t_0 = 0$ & $A(0) = I$, we have $A(t_1) \in \mathcal{U}$. Choose a good partition.

Write A as $A = (A(1)A(t_{m-1})^{-1})(A(t_{m-1})A(t_{m-2})^{-1}) \dots (A(t_2)A(t_1)^{-1})A(t_1)$

Then, we 'define' $\Phi(A)$ by $\Phi(A) = F(A(1)A(t_{m-1})^{-1})F(A(t_{m-1})A(t_{m-2})^{-1})$
$\dots F(A(t_1))$

In last few classes: how to recover Lie gp. info. from the Lie alg.

- BCH thm
- examples: SU(2), SO(3), SU(2)×SU(2), SO(4)

  SU(2), SO(3): isomp. Lie alg. but not isomp as Lie gp.

  SU(2)×SU(2), SO(4): isomp. Lie alg. but not isomp as Lie gp.

- Simply conn. Lie gps with isomp. Lie algs are isomp.

Next: back to studying familiar examples of compact mfg, via studying their 'maximal tori'.

Recall: $U(1) = \{ [e^{i\theta}] : \theta \in [0, 2\pi) \} \simeq SO(2) \simeq S^1$

$U(1)$ is abelian & is path-conn.

Defn: The n-dimensional torus T^n is the gp.

$$T^n = \underbrace{U(1) \times \dots \times U(1)}_{n \text{ copies}}$$

T^n can be viewed as a mfg

$$T^n \simeq \{ \mathrm{diag}(e^{i\theta_1}, \dots, e^{i\theta_n}) : \theta_i \in [0, 2\pi) \} \subseteq GL(n, \mathbb{C})$$

We will show that every compact path-conn. abelian mlg is isomp. to a torus. First some preparation

<u>Def^n</u>: A subset E of a topo. sp. X is said to be <u>discrete</u> if for every $e \in E$, $\exists$ a nbd U of e s.t $U \cap E = \{e\}$

<u>Lem</u>: Let V be a finite-dimensional inner product sp. over $\mathbb{R}$, viewed as a gp. under add^n of vector. Let Γ be a discrete subgp. of V. Then $\exists$ linearly indep. vecs. $v_1, \dots, v_k \in V$ s.t

$$\Gamma = \{m_1 v_1 + \dots + m_k v_k : m_j \in \mathbb{Z} \ \forall j\}$$

eg: $V = \mathbb{R}^2$, $\Gamma = \mathbb{Z}^2$

<u>Pf</u>: Since Γ is discrete, $\exists\, \epsilon > 0$ s.t the only pt. in Γ with $\|\gamma\| < \epsilon$ is $\gamma = 0$

As a result, for any $\gamma, \gamma' \in \Gamma$, if $\|\gamma - \gamma'\| < \epsilon$, then since $\gamma - \gamma'$ is also in Γ, we must have $\gamma = \gamma'$.
It then follows that any b'nd region of V can only contain finitely many pts. of Γ.

If $\Gamma = \{0\}$, the result holds with $k = 0$.

Otherwise $\exists\, 0 \neq \gamma_0 \in \Gamma$ s.t $\|\gamma_0\|$ is minimal among non-zero elems of Γ.

Let W = orthogonal complement of span$\{\gamma_0\}$

Let $P: V \to W$ denote the orthogonal projection

Let $\Gamma' = P(\Gamma)$. Since P is linear, Γ' is a subgp. of W.

<u>Claim</u> : Γ' is discrete in W.

Assume, for contdn, that Γ' is <u>not</u> discrete.

Then $\forall \epsilon > 0$, $\exists\, \delta \neq \delta' \in \Gamma'$ is an elem. with $\|\delta - \delta'\| < \epsilon$.

Then $\exists\, \gamma \in \Gamma$ with $P(\gamma) = \delta - \delta'$.

Hence dist from γ to span$\{\gamma_0\}$ is less than ϵ.

Let β = orthogonal proj of γ onto span$\{\gamma_0\}$

Then β lies b/w $m\gamma_0$ & $(m+1)\gamma_0$ for some $m \in \mathbb{Z}$.

By taking $n = m$ or $n = m+1$, we can assume $\|n\gamma_0 - \beta\| \leq \frac{\|\gamma_0\|}{2}$

As observed before, $\|\beta - \gamma\| < \epsilon$

$\Rightarrow \|\gamma - n\gamma_0\| < \epsilon + \frac{\|\gamma_0\|}{2}$ which is less than $\|\gamma_0\|$ if ϵ is small enough

But then, $0 \neq \gamma - n\gamma_0 \in \Gamma$ is an element with norm less than $\|\gamma_0\|$, contradicting the minimality of $\|\gamma_0\|$.

This completes the proof of the claim.

So, we have W with $\dim W < \dim V$, & $\Gamma' \subset W$ discrete subgp.

We can apply ind^n on the dim. of V.

base case: $\dim V = 0$

Thus, by ind^n hyp., $\exists$ lin. indep. $u_1, \ldots, u_{k-1} \in \Gamma'$ s.t

$$\Gamma' = \{m_1 u_1 + \ldots + m_{k-1} u_{k-1} : m_j \in \mathbb{Z} \ \forall j\}$$

Choose $v_1, \ldots, v_{k-1} \in \Gamma$ s.t $P(v_j) = u_j$

Since $P(v_1), \ldots, P(v_{k-1})$ are lin. indep. in W, it follows that $v_1, \ldots, v_{k-1}, \gamma_0$ are lin. indep. in V.

For any $\gamma \in \Gamma$, $P(\gamma)$ is in Γ', hence, it is of the form $m_1 u_1 + \ldots + m_{k-1} u_{k-1}$

Hence, $\gamma = m_1 v_1 + \ldots + m_{k-1} v_{k-1} + \sigma$ where $\sigma \in \Gamma$ satisfies $P(\sigma) = 0$, i.e $\sigma \in \operatorname{span}\{\gamma_0\}$.

But then, σ must be an integer multiple of γ_0

If not, then we could obtain an elem. of Γ in span$\{\gamma_0\}$ with norm less than $\|\gamma_0\|$, contradicting the choice of γ_0 as an elem. of Γ with minimal norm.

We conclude $\sigma = m_k \gamma_0$ for some $m_k \in \mathbb{Z}$ & hence

$\gamma = m_1 v_1 + \dots + m_{k-1} v_{k-1} + m_k \gamma_0$ as required.

<u>Example</u>: 1) $V = \mathbb{R}^2$, $\Gamma = \mathbb{Z}^2$, generated by $\{e_1, e_2\}$

$$V \longrightarrow T^2$$
$$(t_1, t_2) \longmapsto (e^{2\pi i t_1}, e^{2\pi i t_2})$$

$V/\Gamma \simeq T^2$ isomp. as Lie gps.

2) $V = \mathbb{R}^2$, $\Gamma = \mathbb{Z} \times \{0\}$

$$V/\Gamma \simeq S^1 \times \mathbb{R}$$
$$(t_1, t_2) \mapsto (e^{2\pi i t_1}, t_2)$$

Thm: Any compact, path-conn., abelian mlg T is isomp. to a torus.

Pf: Let $\mathfrak{t}$ be the Lie alg. of T, then $\mathfrak{t}$ is also abelian (proved at the time of defining Lie bracket that if e^{tA} & e^{tB} commute, then $[A,B]=0$)

Hence, $e^A e^B = e^{A+B} \quad \forall\ A, B \in \mathfrak{t}$

Hence, in this case, $\exp : \mathfrak{t} \to T$ is a gp. homomp. when $\mathfrak{t}$ is thought of as a gp. under vector add^n.

Claim: $\exp : \mathfrak{t} \to T$ is surjective

The image $\exp(\mathfrak{t}) \subset T$ contains a nbd V of I in T.

Since here $\exp(\mathfrak{t})$ is a gp., it contains the gp. generated by V.

Since T is path-conn., V generates T.

Hence $\exp(\mathfrak{t})$ contains T, i.e $\exp : \mathfrak{t} \to T$ is surjective.

Next, since $\exp$ is inj. in a nbd of 0, $\Gamma = \ker(\exp)$ must be discrete. By the previous lemma, Γ equals the set of integer linear combinations of some linearly indep. vecs. $v_1, \dots, v_k$ in $\mathfrak{t}$.

Claim: If $\dim \mathfrak{t} = n$, $\mathfrak{t}/\Gamma \simeq T^k \times \mathbb{R}^{n-k}$ gp. isomp. & homeomp.

Pf: Choose $v_{k+1}, \dots, v_n$ s.t $\{v_1, \dots, v_k, v_{k+1}, \dots, v_n\}$ is a basis for $\mathfrak{t}$. Then, define the map

$$\varphi : \mathfrak{t} \to T^k \times \mathbb{R}^{n-k}$$
$$t_1 v_1 + \dots + t_k v_k \mapsto \operatorname{diag}(e^{2\pi i t_1}, \dots, e^{2\pi i t_k}), (t_{k+1}, \dots, t_n)$$

φ is a gp. homomp. with $\ker(\varphi) = \Gamma$ so we conclude that
$\mathfrak{t}/\Gamma \simeq T^k \times \mathbb{R}^{n-k}$

Then exp descends to a bij. homeomp. of $\mathfrak{t}/\Gamma$ with T.

Now, exp is a local diffeomp., hence so is
$\exp : \mathfrak{t}/\Gamma \to T$. Since $\exp : \mathfrak{t}/\Gamma \to T$ is bij., it is a diffeomp.
Since T is compact, it implies that $k = n$, so T is isomp. to $T^n \simeq (S^1)^n$.

Def^n: Let G be a mlg. A torus in G means a subgp. of G which is isomp. to a torus. A maximal torus in G means a torus in G which is not contained in any higher dimensional torus in G.

Note: Every mlg contains at least one maximal torus. Why? Because $\{I\} \subset G$ can be thought of as a 0-dim torus.

If $\{I\}$ is not contained in a 1-dim torus in G, then $\{I\}$ is maximal. Otherwise, choose some 1-dim torus T^1 in G. If T^1 is not contained in any 2-dim torus in G, then T^1 is a maximal torus in G.

Otherwise choose a 2-dim torus T^2 in G (containing T^1), etc. This process must terminate, because G cannot contain a torus of dim higher than the dim of G.

Ppⁿ: If $R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, then

$$T' = \left\{ \underbrace{\mathrm{diag}(R_\theta, 1)}_{R_\theta'} = \begin{bmatrix} R_\theta & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} : \theta \in [0, 2\pi) \right\}$$

is a maximal torus in $SO(3)$.

Pf: Note that all matrices in T' are all rotⁿs of (e_1, e_2) plane through some angle θ leaving e_3 axis fixed.
Suppose that T is any torus in $SO(3)$ that contains T'.
Then T is abelian, so any $A \in T$ commutes with all $R_\theta' \in T'$.

We will show that if $AR_\theta' = R_\theta' A \ \forall R_\theta' \in T'$, then $A \in T'$.
so $T = T'$ & hence T' is maximal.
So, let $A \in SO(3)$.

Claim: $Ae_1, Ae_2 \in (e_1, e_2)$ plane

Pf: Suppose $Ae_1 = a_1e_1 + a_2e_2 + a_3e_3$

Since A commutes with all $R_\theta' \in T'$, in particular with $R_\pi'^{-1} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$,

we have $AR_\pi'(e_1) = A(-e_1) = -a_1e_1 - a_2e_2 - a_3e_3$

& $R_\pi' A(e_1) = R_\pi'(a_1e_1 + a_2e_2 + a_3e_3) = -a_1e_1 - a_2e_2 + a_3e_3$

$\Rightarrow a_3 = 0$, hence $Ae_1 \in \text{span}\{e_1, e_2\}$. A similar argument shows that also $Ae_2 \in \text{span}\{e_1, e_2\}$

As a result of this claim, A is an isometry of the (e_1, e_2) plane that fixes 0. Hence $A = \text{diag}(B, 1)$ with $B \in SO(2)$ or $A = \text{diag}(B, -1)$ with $B \in O(2) \setminus SO(2)$

[Recall, we showed earlier in the course that

1) $O(n)$ acts on $\mathbb{R}^n$ by isometries
2) Any isometry of $\mathbb{R}^n$ which fixes the origin is given by T_X, where $X \in O(n)$]

Case 1: If $B \in SO(2)$, then $A = \text{diag}(B, 1) \in T'$.

Case 2: $B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} B'$, $B = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix} \in SO(2)$

i.e $B = \begin{bmatrix} -\cos\theta_1 & \sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix}$

But then $BR_\theta \neq R_\theta B$ when $0 < \theta < 2\pi$

So, the second case does not occur.

Example

1. If $R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, then

$$T = \left\{ \operatorname{diag}(R_{\theta_1}, \dots, R_{\theta_m}) = \begin{bmatrix} R_{\theta_1} & & 0 \\ & \ddots & \\ 0 & & R_{\theta_m} \end{bmatrix},\ \theta_i \in [0, 2\pi) \right\}$$

$\subseteq SO(2m)$

T is a torus

2. $T = \{ \operatorname{diag}(R_{\theta_1}, \dots, R_{\theta_m}, 1) : \theta_i \in [0, 2\pi) \} \subseteq SO(2m+1)$

T is a torus

3. $T = \{ \operatorname{diag}(e^{i\theta_1}, \dots, e^{i\theta_n}) : \theta_j \in [0, 2\pi) \} \subset U(n)$

T is a torus

4. $T = \{ \operatorname{diag}(e^{i\theta_1}, \dots, e^{i\theta_{n-1}}, e^{-i(\theta_1 + \dots + \theta_{n-1})}) : \theta_j \in [0, 2\pi) \} \subset SU(n)$

T is a torus

Thm: Each of the examples 1-4 above, is a maximal torus in the resp. mlg.

Pf: We need to prove that in each case, T is not contained in a higher dimensional torus of the gp. G.
In each case, we will justify this by proving that any elm. of G which commutes with all elms. of T, must lie in T.

Case 1: $G = SO(2m)$
Let $e_1, \ldots, e_{2m}$ be the std. basis of $\mathbb{R}^{2m}$.
Let $R_{\theta_1, \ldots, \theta_m} = \mathrm{diag}(R_{\theta_1}, \ldots, R_{\theta_m})$. Then, $R_{\theta_1, \ldots, \theta_m}$ is the product of matrices, each of which is a rot^n in one 2-plane, keeping fixed the basis vecs. orthogonal to that 2-plane.

- Rot^n of the (e_1, e_2) plane through angle θ_1 : $R_{\theta_1, 0, \ldots, 0}$
- Rot^n of the (e_3, e_4) plane through angle θ_1 : $R_{0, \theta_2, \ldots, 0}$

$\vdots$

- Rot^n of the (e_{2m-1}, e_{2m}) plane through angle θ_m: $R_{0, 0, \ldots, \theta_m}$

Suppose $A \in SO(2m)$ commutes with each $R_{\theta_1, \ldots, \theta_m}$.
We will show that

- Ae_1, Ae_2 lie in (e_1, e_2) plane
- Ae_3, Ae_4 lie in (e_3, e_4) plane

  $\vdots$

- Ae_{2m-1}, Ae_{2m} lie in (e_{2m-1}, e_{2m}) plane

Once we show this, it will follow that A is a product of rot^n's of these planes, hence A is an elem. of the given torus T.

[Note that there is a possibility that the restriction of A to one of these 2-planes P, is <u>not</u> a rot^n but is a product of rot^n & a reflection. But this can be ruled out by the fact that A commutes with all $R_{\theta_1, \dots, \theta_m}$, including those that rotate only the plane P. Then it can be shown as in the case of $SO(2)$, that A rotates P.]

So, we will show that $Ae_1 \in (e_1, e_2)$ plane, the other cases are similar.

Suppose that $AR_{\theta_1, \dots, \theta_m} = R_{\theta_1, \dots, \theta_m} A \quad \forall \, R_{\theta_1, \dots, \theta_m} \in T$

In particular, $AR_{\pi, 0, \dots, 0}(e_1) = R_{\pi, 0, \dots, 0} A(e_1)$

Then if $Ae_1 = a_1e_1 + a_2e_2 + \dots + a_{2m}e_{2m}$

$$AR_{\pi,0,\dots,0}(e_1) = A(-e_1)$$

$$= -a_1e_1 - a_2e_2 \cdots - a_{2m}e_{2m}$$

but

$$R_{\pi,0,\dots,0}A(e_1) = R_{\pi,0,\dots,0}(a_1e_1 + a_2e_2 + \dots + a_{2m}e_{2m})$$

$$= -a_1e_1 - a_2e_2 + a_3e_3 + \dots + a_{2m}e_{2m}$$

Hence, $a_3 = a_4 = \dots = a_{2m} = 0$, as req.

The argument is similar for any other e_k. Hence $A \in T$ as claimed.

<u>Case 2</u>: $G = SO(2m+1)$

We can generalize the argument for $SO(3)$ from the prev. pp^n, using maps such as $R'_{\pi,0,\dots,0} = \mathrm{diag}(R_\pi, R_0, \dots, R_0, 1)$ in place of R_π'.

Case 3 : $G = U(n)$.

Let $e_1, \dots, e_n$ be the std. basis for $\mathbb{C}^n$ (as a $\mathbb{C}$-vec. sp.)

Let $z_{\theta_1, \dots, \theta_n} = \mathrm{diag}(e^{i\theta_1}, \dots, e^{i\theta_n})$. Suppose $A \in U(n)$ s.t. A commutes with each elem of T. In particular, A commutes with

$z_{\pi, 0, \dots, 0} = \mathrm{diag}(-1, 1, \dots, 1)$.

Then if $Ae_1 = a_1 e_1 + \dots + a_n e_n$, we have

$$A z_{\pi, 0, \dots, 0}(e_1) = A(-e_1) = -a_1 e_1 - a_2 e_2 - \dots - a_n e_n$$

$$\& \quad z_{\pi, 0, \dots, 0} A(e_1) = -a_1 e_1 + a_2 e_2 + \dots + a_n e_n$$

Hence, $a_2 = a_3 = \dots = a_n = 0$

Thus $A_1 e_1 = c_1 e_1$ for some $c_1 \in \mathbb{C}$. Sim., we obtain $Ae_k = c_k e_k$ $\forall k$.

Also, since $A \in U(n)$, we know that $\{Ae_1, \dots, Ae_n\}$ is an o.n.b.

Hence each $|c_k| = 1$, so $c_k = e^{i\varphi_k}$, hence $A = R_{\varphi_1, \dots, \varphi_n}$, so $A \in T$.

Case 4: $G = SU(n)$

For $n > 2$, we can argue as for $U(n)$, except that we need to use the fact that A commutes with both $z_{\pi,\pi,0,\ldots,0}$ & $z_{\pi,0,\pi,0,\ldots,0}$ to conclude that $Ae_1 = c_1 e_1$.

This is because $z_{\pi,0,\ldots,0}$ is <u>not</u> in $SU(n)$ since it has det -1.

For $n = 2$, we argue as follows

Suppose $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU(n)$ commutes with each $z_{\theta,-\theta}$ in T.
In particular, A commutes with $z_{\frac{\pi}{2},-\frac{\pi}{2}} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, which implies that $\begin{pmatrix} ai & -bi \\ ci & -di \end{pmatrix} = \begin{pmatrix} ai & bi \\ -ci & -di \end{pmatrix}$

Hence, $b = c = 0$, so $A = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in T$

<u>Thm</u>: The centres of these groups are

1. If $m \geq 2$, $Z(SO(2m)) = \{\pm I\}$,

$Z(SO(2)) = SO(2)$

2. $Z(SO(2m+1)) = \{I\}$
3. $Z(U(n)) = \{\omega I : |\omega| = 1\}$
4. $Z(SU(n)) = \{\omega I : \omega^n = 1\}$

<u>Pf</u>: The arguments in the prev. thm show that any elem. in G ($G = SO(n), U(n), SU(n)$) which commutes with all elems. of the given maximal torus T, is actually an elem. of T.

If $A \in Z(G)$, then A commutes with all elems of G, in particular, A commutes with all elems in T, & hence A must be an elem. of T.

<u>Case 1</u>: $G = SO(2m)$, $m \geq 2$

$A \in Z(SO(2m)) \Rightarrow A = R_{\theta_1 \dots \theta_m}$ & A commutes with all elems of $SO(2m)$.

Note that $R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ does <u>not</u> commute with the matrix

$I_2^* = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ unless $\sin\theta = 0$ (& hence $\cos\theta = \pm 1$)

T'fore, if we build a matrix $I^*_{2m} \in SO(2m)$, with copies of I^*_2 on the diagonal, then $R_{\theta_1 \cdots \theta_n}$ will commute with I^*_{2m} only if each $\sin\theta_k = 0$ & $\cos\theta_k = \pm 1$.

Hence $A \in Z(SO(2m))$ implies A is diagonal, with diagonal entries ± 1. If both $+1$ & -1 occur, we can find a matrix in $SO(2m)$ which does not commute with A (namely, a matrix with R_θ on the diagonal at the position of the adjacent $+1$ & -1s elsewhere on the diagonal & 0s everywhere else).

Thus either $A = I$ or $A = -I$

Note that $I, -I \in Z(SO(2m))$. Hence $Z(SO(2m)) = \{I, -I\}$

Case 2 : $G = SO(2m+1)$

The argument is similar to case 1, except for the last step. The $(2m+1) \times (2m+1)$ matrix $-I$ does not belong in $SO(2m+1)$, since it has $\det -1$.

Case 3: $G = U(n)$

If $A \in Z(U(n))$, then $A = z_{\theta_1 \cdots \theta_n}$ for some $\theta_1, \ldots, \theta_n$ &

A commutes with all elems. of $U(n)$.

If $n=1$, $U(1)$ is abelian, hence $Z(U(1)) = U(1)$.

If $n \geq 2$, note that $\begin{bmatrix} e^{i\theta_1} & 0 \\ 0 & e^{i\theta_2} \end{bmatrix}$ commutes with $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ iff

$e^{i\theta_1} = e^{i\theta_2}$. Thus, since A commutes with the elem

$$G_{12} = \begin{bmatrix} 0 & 1 & & 0 \\ 1 & 0 & & \\ & & 1 & \\ & & & \ddots & \\ 0 & & & & 1 \end{bmatrix}$$

of $U(n)$, we must have $e^{i\theta_1} = e^{i\theta_2}$.

Sim., by using appropriate elems. of $U(n)$, we can see that

$e^{i\theta_1} = \cdots = e^{i\theta_n}$. Hence $A \in Z(U(n))$ implies $A = e^{i\theta} I$ for some θ.

Conversely, all matrices of the form $e^{i\theta} I$ commute with everything

(& they are in $U(n)$)

Hence, $Z(U(n)) = \{ e^{i\theta} I : \theta \in \mathbb{R} \}$

$= \{ \omega I : |\omega| = 1 \}$

Case 4: $G = SU(n)$

An argument similar to $U(n)$ shows that $A \in Z(SU(n))$ implies A has the form ωI with $|\omega| = 1$.

Since $A = \omega I \in SU(n)$, we also have $1 = \det A = \det(\omega I) = \omega^n$

Conversely, any matrix of the form ωI with $\omega^n = I$, belongs in $SU(n)$ & commutes with all elms. of $SU(n)$.

Hence $Z(SU(n)) = \{\omega I : \omega^n = 1\}$

Example: As an application of this thm, we see that

- $SU(2)$ is not isomp. to $SO(3)$
- $SU(2) \times SU(2)$ is not isomp. to $SO(4)$

since their centres are not isomp.